ADDITIVE SUMMABLE PROCESSES AND THEIR STOCHASTIC INTEGRAL

By

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I dedicate this work to my husband Omar, my parents Marius and Angela, my sister Mihaela, my grandmother Ecaterina and all my good friends.

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TABLE OF CONTENTS

page

ACKNOWLEDGMENTS iii

ABSTRACT vi

INTRODUCTION 1

CHAPTER

1 NOTATIONS AND DEFINITIONS 4

- 1.1 The Variation 4
- 1.2 The Semivariation 5
- 1.3 The Canonical Extension 8
- 1.4 The Space $\mathcal{F}_{\mathcal{D}}(m)$ 8
- 1.5 The Immediate Integral 12

2 ADDITIVE MEASURES 13

- 2.1 Extensions of Additive Measures 13
- 2.2 The Lebesgue–Stieltjes Integral for Functions with Finite Semivariation 15
- 2.3 Optional and Predictable Measures 19
- 2.4 Dual Projections of Measures 22

3 ADDITIVE SUMMABLE PROCESSES 25

- 3.1 The Measure I_X 25
- 3.2 The Measures $(I_X)_z$ 26
- 3.3 Additive Summable Processes 26
- 3.4 The Space $\mathcal{F}_D(\tilde{I}_{F,L^p_G})$ 27
- 3.5 The Integral $\int H dI_X$ 28
- 3.6 Lebesgue Convergence for the Integral $\int H dI_X$ 28

4 THE STOCHASTIC INTEGRAL 30

- 4.1 Definition of the Stochastic Integral $H \cdot X$ 30
- 4.2 The Stochastic Integral and Stopping Times 34
- 4.3 The Integral $\int H dI_{X^T}$ 46
- 4.4 Convergence Theorems 49

4.5 Summability of the Stochastic Integral 54
4.6 Summability Criterion 59

5 EXAMPLES OF ADDITIVE SUMMABLE PROCESSES 65

5.1 Examples of Additive Summable Processes 65
5.2 Processes with Integrable Semivariation 66
5.3 Example of a Summable Process That is Not a Semimartingale 70

REFERENCES 77

BIOGRAPHICAL SKETCH 79

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In this dissertation we define and study a class of summable processes that is larger than the classes other researchers have worked on, called additive summable processes.

We relax the definition of a summable processes $X : \Omega \times \mathbb{R}_+ \to E \subset L(F, G)$ by asking for the associated measure I_X to have just an additive extension to the predictable $\sigma-$algebra $\mathcal{P}$, such that each of the measures $(I_X)_z$, for $z \in (L^p_G)^*$, being $\sigma-$additive, rather than having a $\sigma-$additive extension. We define a stochastic integral with respect to such a process and we prove several properties of the integral. After that we show that this class of summable processes contains all processes $X : \Omega \times \mathbb{R}_+ \to E \subset L(F, G)$ with integrable semivariation if $c_0 \not\subset G$. Finally we prove that the stochastic integral defined by Dinculeanu in 2000 (hence this one also) is more comprehensive than the classical stochastic integral.

INTRODUCTION

In this work we study the stochastic integral in the case of Banach-valued processes from a measure-theoretical point of view.

The classical stochastic integration (for real-valued processes) refers only to integrals with respect to semimartingale (Dellacherie and Meyer [DM78]). A similar technique has also been applied by Kunita [Kun70], for Hilbert valued processes, making use of the inner product. A number of technical difficulties emerge for Banach spaces, since the Banach space lacks an inner product. Kussmaul [Kus77] defined a measure-theoretical stochastic integral but only for real-valued processes.

Vector integration using different approaches were presented in several books by Dinculeanu [Din00], Diestel and Uhl [DU77], and Kussmaul [Kus77]. Brooks and Dinculeanu [BD76] were the first to present a version of integration with respect to a vector measure with finite semivariation. Later, the same authors [BD90] presented a stochastic integral with respect to so-called summable Banach-valued processes.

A Banach-valued process X is called *summable* if the Doleans-Dade measure I_X defined on the ring generated by the predictable rectangles can be extended to a σ-additive measure with finite semivariation on the corresponding σ-algebra $\mathcal{P}$. The summable process X plays the role of the square integrable martingale in the classical theory: a stochastic integral $H \cdot X$ can be defined with respect to X as a cadlag modification of the process $\left(\int_{[o,t]} H\, dI_X\right)_{t\geq 0}$ of integrals with respect to I_X such that $\int_{[0,t]} H dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$.

In [Din00] Dinculeanu presents a detailed account of the integration theory with respect to these summable processes, from a measure-theoretical point of view.

Our attention turned to a further generalization of the stochastic integral. Besides the processes considered in the classical theory (Hilbert-valued square-integrable martingales and processes with integrable variation), the class of summable processes also includes processes with integrable *semivariation*, as long as the Banach space E satisfies some restrictions. To get rid of some of these restrictions, we redefine, in Chapter 4, the notion of summability: now we only require that I_X can be extended to an *additive* measure on $\mathcal{P}$, but such that each of the measures $(I_X)_z$, for $z \in Z$ a norming space for L^p_G, is σ-additive. With this new notion of summability, called additive summability, the stochastic integral is then defined, in Section 5.1, as before. The rest of Chapter 5 is dedicated to proving the same type of properties of the stochastic integral as in Dinculeanu [Din00], namely measure theoretical properties.

In doing that, we encounter two types of problems: extension problems and convergence problems. Those problems arose because in this case the measure I_X is required to have an additive extension rather than an $\sigma-$additive one. To overpass them we added Chapter 3 to our work, a chapter that could be viewed as independent of the rest of the work and that deals with properties of additive measures. Bongiorno and Dinculeanu [BD01] proved the existence and uniqueness of an extension called "canonical extension" (for additive measures) but the range of the extension is different than the range of the initial measure. In Section 3.1 we prove that if $c_0 \not\subset G$ this extension has the same range. In Section 3.2 we prove another extension theorem while Section 3.3 and 3.4 are dedicated to dual projection of additive measures.

The first properties we prove in Chapter 5 are regarding the stopping times and the integral. We observe that additive summable processes have fewer properties than do the summable processes. If X is an additive summable process and T a stopping time, X^T is additive summable only if T is simple. If T is not a simple stopping time, then X^T may not be additive summable; but even in this case we can define a stochastic integral $H \cdot X^T$ and if H is X-integrable and X^T-integrable then the associative formula $(H \cdot X)^T = H \cdot X^T = (1_{[0,T]}H) \cdot X$ holds.

In Section 5.4 we prove that the space of integrable processes is complete and that uniform convergence theorem, Vitali and Lebesgue type theorems hold for this class of integrable processes.

In Section 5.5 we prove that, for vector-valued integrable processes, the measure $I_{H \cdot X}$ is $\sigma-$additive, even if the measure I_X is just additive, and that the stochastic integral is summable.

In Section 5.6 we present a summability criterion that gives a relation between the boundedness of the measure I_X on $\mathcal{R}$ and the existence of an additive canonical extension to $\mathcal{P}$.

Chapter 6 consists of examples of additive summable processes. While processes with finite variation and square integrable martingales are obvious examples of additive summable processes (since they are summable), we prove in Section 6.2 that if $c_0 \not\subset G$ then the processes with finite semivariation are also additive summable. Hence the class of additive summable processes is larger than the class of summable processes, since the processes with finite semivariation are summable iff $c_0 \not\subset E$ and $c_0 \not\subset G$. Finally, in Section 6.3, we establish that the stochastic integral defined by Dinculeanu [Din00], hence ours also, is indeed more comprehensive than the classical integral (which is defined with respect to a semimartingale), by proving that if certain conditions are satisfied by a Banach space E, then there exist E-valued summable processes that are not semimartingales. To show that X is not a semimartingale we proved a theorem that does not involve summability or the stochastic integral definition: in certain conditions, a semimartingale with finite semivariation has locally integrable variation and semivariation.

CHAPTER 1
NOTATIONS AND DEFINITIONS

In this chapter we present several notations and definitions used in this work. Also, we state properties of the notions defined.

The framework for this section is the following:

- E, F, G are Banach spaces with $E \subset L(F, G)$ continuously, that is, $|x(y)| \leq |x||y|$ for $x \in E$ and $y \in F$; for example, $E = L(\mathbb{R}, E)$.
- If M is any Banach space, we denote by $|x|$ the norm of an element $x \in M$, by M_1 its unit ball of M and by M^* the dual of M.
- A space $Z \subset G^*$ is called a *norming space for* G, if for every $x \in G$ we have

$$|x| = \sup_{z \in Z_1} |\langle x, z \rangle|.$$

1.1 The Variation

Through the rest of this chapter, S is a set and $\mathcal{R}$, $\mathcal{D}$, $\mathcal{S}$, Σ are respectively a ring, a $\delta-$ring, a $\sigma-$ring, and a $\sigma-$algebra of subsets of S. Also E is a Banach space and $m : \mathcal{R} \to E$ an additive measure. For any class $\mathcal{C}$ of subsets of S we denote by $\mathcal{C}_{\text{loc}}$ the class of sets $A \subset S$ that are "locally" in $\mathcal{C}$, that is, such that $A \cap B \in \mathcal{C}$ for every $B \in \mathcal{C}$. If $\mathcal{R}$ is a ring, then $\mathcal{R}_{\text{loc}}$ is an algebra.

Definition 1. For every set $A \subset S$ (not necessarily from $\mathcal{R}$ or $\mathcal{R}_{\text{loc}}$) the variation of m on A is a number, finite or $+\infty$, denoted by $var(m, A)$ or $\overline{m}(A)$ and defined by the following equality:

$$var(m, A) = \sup \sum |m(A_i)|,$$

the supremum being taken for all finite families $(A_i)_{i \in I}$ of mutually disjoint sets from $\mathcal{R}$ contained in A.

The above definition can be stated for any set function m (not necessarily additive) defined on any class $\mathcal{C}$ containing ϕ (not necessarily a ring), such that $m(\phi) = 0$.

For $A \in \mathcal{R}_{\text{loc}}$ we usually denote $|m|(A) = var(m, A)$.

If $A \in \mathcal{R}$, in the definition of $|m|(A)$ we can take only families $(A_i)_{i \in I}$ of disjoint sets from $\mathcal{R}$ with union *equal* to A.

We say that m has *finite* (respectively *bounded*) variation, if $|m|(A) < \infty$ for every $A \in \mathcal{R}$ (respectively if $|m|(S) < \infty$).

Let $\mathcal{D}$ be the $\delta-$ring generated by $\mathcal{R}$. If $m' \colon \mathcal{D} \to E$ is an extension of m, then for the definition of $|m'|$ we consider families $(A_i)_{i \in I}$ of sets from $\mathcal{D}$. Consequently, for a set $A \subset S$ we have $|m|(A) \leq |m'|(A)$ and the inequality may be strict. We have equality for $A \in \mathcal{R}$ if m' is σ-additive and has finite variation (Theorem 2.11 in [Din00]). For a more detailed account of the variation see [Din00], § 2.

1.2 The Semivariation

The framework for this section is a ring $\mathcal{R}$ of subsets of S, three Banach spaces E, F, G such that $E \subset L(F, G)$ continuously and an additive measure $m : \mathcal{R} \to E \subset L(F, G)$.

Definition 2. For every set $A \subset S$ (not necessarily in $\mathcal{R}$ or $\mathcal{R}_{\text{loc}}$) the semivariation of m on A relative to the embedding $E \subset L(F, G)$ (or relative to the pair (F, G)) is a number, finite or $+\infty$, denoted $svar_{F,G}(m, A)$ or $\tilde{m}_{F,G}(A)$ and defined by the equality

$$\tilde{m}_{F,G}(A) = \sup |\sum_{i \in I} m(A_i) x_i|,$$

where the supremum is taken for all finite families $(A_i)_{i \in I}$ of disjoint sets from $\mathcal{R}$ contained in A and all families $(x_i)_{i \in I}$ of elements from F_1.

The above definition can be stated for any set function m defined on any class $\mathcal{C}$ containing ϕ, such that $m(\phi) = 0$.

If $A \in \mathcal{R}$, in the definition of $\tilde{m}_{F,G}(A)$ we can restrict ourselves to the families $(A_i)_{i\in I}$ of disjoint sets from $\mathcal{R}$ with union equal to A.

The semivariation $\tilde{m}_{F,G}$ depends only on the norms of the space F and G, but not on the norm of E.

For the embedding $E = L(\mathbb{R}, E)$ we sometimes write *svar* m or $\tilde{m}$ or $\|m\|$ for $\tilde{m}_{\mathbb{R},E}$. We agree also to write $\tilde{m}$ instead of $\tilde{m}_{F,G}$, if the pair (F, G) is understood.

We say that m has finite (respectively bounded) semivariation relative to (F, G) if $\tilde{m}_{F,G}(A) < \infty$ for every $A \in \mathcal{R}$ (respectively $\tilde{m}_{F,G}(S) < \infty$).

From 2 we deduce that for any set $A \subset S$, the semivariation $\tilde{m}_{F,G}(A)$ can also be defined by the equality

$$\tilde{m}_{F,G}(A) = \sup |\int sdm|,$$

where the supremum is taken for all $\mathcal{R}-$step functions $s : S \to F$ with $|s| \leq \varphi_A$.

We state now some properties of the semivariation.

- If $\mathcal{P}$ is a semiring generating the ring $\mathcal{R}$, then in the definition of the semivariation $\tilde{m}_{F,G}(A)$ we can take only families (A_i) of disjoint sets from $\mathcal{P}$ contained in A.

If $m' : \mathcal{P} \to E$ is the restriction of m to $\mathcal{P}$, this means that

$$svar_{F,G}(m', A) = svar_{F,G}(m, A).$$

- If the embedding $E \subset L(F, G)$ is an isometry then

$$|m(A)| \leq \tilde{m}_{F,G}(A), \ \textit{for}\ A \in \mathcal{R}.$$

- $\tilde{m}_{F,G}(A) \leq |m|(A)$, for every $A \subset S$.

- If $\mathcal{R}'$ is a subring of $\mathcal{R}$ and if m' is the restriction of m to $\mathcal{R}'$, then for every set $A \subset S$ we have

$$svar_{F,G}(m', A) \leq svar_{F,G}(m, A)$$

Proposition 3. *The set function $\tilde{m}_{F,G}$ is finitely subadditive on $\mathcal{R}_{loc}$.*

If m *is* $\sigma-$*additive on* $\mathcal{R}$, *then for any sequence* (A_n) *of disjoint sets from* $\mathcal{R}_{loc}$ *with union* A *not necessarily in* $\mathcal{R}_{loc}$ *we have* $\tilde{m}_{F,G}(A) \leq \sum_n \tilde{m}_{F,G}(A_n)$.

If $G = \mathbb{R}$, the semivariation and the variation are equal:

Proposition 4. *Assume* $E \subset L(F, \mathbb{R})$ *isometrically. If* $m : \mathcal{R} \to E \subset L(F, \mathbb{R})$ *is an additive measure, then* $\tilde{m}_{F,\mathbb{R}} = \overline{m}$.

Proposition 5. *If* $m : \mathcal{R} \to E \subset L(F, G)$ *is an additive measure, then*

$$\tilde{m}_{F,G} \leq \tilde{m}_{E^*,\mathbb{R}} = \overline{m}.$$

If the embedding $E \subset L(F, G)$ *is an isometry, then*

$$\tilde{m}_{\mathbb{R},E} \leq \tilde{m}_{F,G}.$$

For a more detailed account of the semivariation see [Din00], § 4.

One of the most important properties of the semivariation is that it can be computed by means of a family of positive measures obtained in connection with a norming space for G.

We assume that $m : \mathcal{R} \to E \subset L(F, G)$ is an additive measure on a ring $\mathcal{R}$, and that $Z \subset G^*$ is a space norming for G. In particular, we can take $Z = G^*$.

For each $z \in Z$ we define the set function $m_z : \mathcal{R} \to F^*$ by the equality

$$\langle x, m_z(A) \rangle = \langle m(A)x, z \rangle, \text{ for } x \in F \text{ and } A \in \mathcal{R}.$$

Then m_z is an additive measure.

We denote by $|m_z|$ the restriction of the variation $\overline{m}_z$ to $\mathcal{R}_{\text{loc}}$.

The semivariation $\tilde{m}_{F,G}$ can be computed by means of the variations $\overline{m}_z$.

Proposition 6. *For any space* $Z \subset G^*$ *norming for* G *we have*

$$\tilde{m}_{F,G} = \sup_{z \in Z_1} \overline{m}_z.$$

Proposition 7. *Let $m : \mathcal{R} \to E \subset L(F, G)$ be an additive measure defined on a ring $\mathcal{R}$ and let Z be a closed subspace of G^*, norming for G. For any set $A \subset S$ we have*

$$\tilde{m}_{F,G}(A) < \infty \text{ iff } \overline{m}_z(A) < \infty, \text{ for every } z \in Z.$$

1.3 The Canonical Extension

Next we define the canonical extension of a $\sigma-$additive measure. Consider a $\sigma-$additive measure $m : \mathcal{R} \to [0, \infty]$. Using the Carathéodory procedure, m is extended to a $\sigma-$additive measure $m' : \mathcal{S} \to [0, \infty]$ on the $\sigma-$ring $\mathcal{S}$ generated by $\mathcal{R}$ such that $\sup_{A\in\mathcal{R}} m(A) = \sup_{A\in\mathcal{S}} m'(A)$. In the case that $\mathcal{S}$ is not a $\sigma-$algebra we obtain a second extension $m'' : \Sigma \to [0, \infty]$ by

$$m''(A) = \sup\{m(B) : B \in \mathcal{S}, B \subset A\}, \text{ for } A \in \Sigma.$$

In this case we also have $\sup_{A\in\mathcal{R}} m(A) = \sup_{A\in\Sigma} m''(A)$. If m is $\sigma-$finite on $\mathcal{R}$ then the extension m' of the measures m to $\mathcal{S}$, is unique; but in general m' is not necessarily unique. We note that m'' is the smallest extension of m' from $\mathcal{S}$ to Σ. We call m'' the canonical extension of the positive measure m.

1.4 The Space $\mathcal{F}_{\mathcal{D}}(m)$

The framework for this section is a $\delta-$ring $\mathcal{D}$, a Banach space D, and an additive measure $m : \mathcal{D} \to E \subset L(F, G)$ with finite semivariation and $Z \subset G^*$ a norming space for G.. We assume that for each $z \in G^*$, the measure $m_z : \mathcal{D} \to F^*$ is $\sigma-$additive.

In this paragraph we define the seminorm $\tilde{m}_{F,G}(f)$ for $m-$measurable functions $f : S \to F$, the space $\mathcal{F}_{F,G}(m)$ of functions f with $\tilde{m}_{F,G}(f) < \infty$ and then the integral $\int f dm \in Z^*$ for functions $f \in \mathcal{F}_{F,G}(m)$. For a more detailed account of this part see [Din00], §5. Also we present several inclusion properties for spaces of type $\mathcal{F}_{\mathcal{D}}(m)$. But first we need to give several definitions:

- We say a set $A \in \Sigma$ is $m-$negligible if for every set $B \in \mathcal{D}$ with $B \subset A$ we have $m(B) = 0$. Hence a set $A \in \Sigma$ is $m-$negligible iff $|m|(A) = 0$ or iff $\tilde{m}_{F,G}(A) = 0$

for any embedding $E \subset L(F,G)$. We deduce that $A \in \Sigma$ is $m-$negligible iff A is m_z-negligible for every $z \in Z$.

- We say that the set $A \subset S$ is $m-$negligible iff it is contained in an $m-$negligible set $B \in \Sigma$.

- We say that a property is true $m-$almost everywhere if it is true outside of an $m-$negligible set.

- Let (S, Σ, m) be a measure space. A function $f : S \to F$ or $\overline{\mathbb{R}}$ is said to be $m-$measurable if it is equal $m-$a.e. to a Σ-measurable function.

Definition 8. For every function $f : S \to D$ we define

$$\tilde{m}_{F,G}(f) = \tilde{m}_{F,G}(|f|) = \sup |\int sdm|$$

where the supremum is taken for all $\mathcal{D}-$step functions $s : S \to F$ with $|s| \leq |f|$.

The next proposition gives a computation for $\tilde{m}_{F,G}(f)$ if f is $m-$measurable.

Proposition 9. *If $f : S \to D$ is $m-$measurable, then*

$$\tilde{m}_{F,G}(f) = \sup\{\int |f|d|m_z| : z \in Z_1\}.$$

Now we define the $m-$integrable functions.

Definition 10. We denote by $\mathcal{F}_D(\tilde{m}_{F,G})$ or $\mathcal{F}_D(m_{F,G})$ the set of all $m-$measurable functions $f : S \to D$ with $\tilde{m}_{F,G}(f) < \infty$ and we call it the space of $D-$valued, $m_{F,G}-$integrable function.

If the spaces F, G are understood we shall write $\mathcal{F}_D(\tilde{m})$ instead of $\mathcal{F}_D(\tilde{m}_{F,G})$.

Remark 11. By the previous proposition for each function $f \in \mathcal{F}_D(\tilde{m})$ we have $\int |f|d|m_z| \leq \tilde{m}(f)$ for $z \in Z_1$. Hence we deduce that

$$\mathcal{F}_D(\tilde{m}) \subset L^1_D(|m_z|), \text{ for every } z \in Z,$$

or

$$\mathcal{F}_D(\tilde{m}) \subset \bigcap_{z \in Z} L^1_D(|m_z|).$$

Next we state a proposition which is an extension of Proposition 7.

Proposition 12. *Assume that* Z *is closed in* G^*. *Let* $f : S \to D$ *be an* $m-$*measurable function. Then*

$$\tilde{m}(f) < \infty \text{ iff } \int |f| d|m_z| < \infty, \text{ for every } z \in Z.$$

Note now that the converse inclusion in Remark 11 is also true if the norming space Z is closed in G^*. Hence we can state the following corollary:

Corollary 13. *If* Z *is closed in* G^*, *then*

$$\mathcal{F}_D(\tilde{m}) = \bigcap_{z \in Z} L^1_D(|m_z|).$$

Next we define the integral $\int f dm$.

Assume that $D = F$. To simplify the notation, we write $\mathcal{F}_{F,G}(m)$ or $\mathcal{F}_{F,G}(\tilde{m})$ instead of $\mathcal{F}_F(\tilde{m}_{F,G})$.

Let $f \in \mathcal{F}_{F,G}(m)$. Then, by Corollary 13, we have $f \in L^1_F(|m_z|)$ for every $z \in Z$, hence, the integral $\int f dm_z$ is defined, and it is a real number. Therefore the mapping $z \mapsto \int f dm_z$ is a continuous linear functional on Z hence it belongs to Z^*. We denote it by $\int f dm$ and we call it the integral of f with respect to m.

Next we state and prove several theorems regarding properties of $\mathcal{F}_{\mathcal{D}}(m)$-type spaces.

Proposition 14. *Let* $x \in F$ *and define the measure* $mx : \mathcal{D} \to G$ *by*

$$(mx)(A) = m(A)x, \text{ for } A \in \mathcal{D}.$$

Then

a) *For every* $z \in Z$, *the measure* $(mx)_z$ *is* $\sigma-$*additive,* mx *has finite semivariation* $(\widetilde{mx})_{\mathbb{R},G}$ *and for every positive,* $\Sigma-$*measurable function* $\varphi : S \to \mathbb{R}_+$ *we have*

$$(\widetilde{mx})_{\mathbb{R},G}(\varphi) \leq |x| \tilde{m}_{F,G}(\varphi)$$

and

$$\tilde{m}_{F,G}(\varphi x) = |x|\tilde{m}_{F,G}(\varphi).$$

If $Z = G^*$, *then* mx *is* $\sigma-$*additive.*

b) $\mathcal{F}_{\mathbb{R}}(\tilde{m}_{F,G}) \subset \mathcal{F}_{\mathbb{R}}((\widetilde{mx})_{\mathbb{R},G})$ *and* $x\mathcal{F}_{\mathbb{R}}(\tilde{m}_{F,G}) \subset \mathcal{F}_F(\tilde{m}_{F,G})$

c) *Assume* $S = \bigcup\limits_{n\in\mathbb{N}} S_n$ *with* $S_n \in \mathcal{D}$. *If* $\varphi \in \mathcal{F}_{\mathbb{R}}(\tilde{m}_{F,G})$, *then*

$$\int \varphi x dm = \int \varphi d(mx)$$

d) *Assume* $S = \bigcup\limits_{n\in N} S_n$ *with* $S_n \in \mathcal{D}$. *If* $\varphi : S \to \mathbb{R}$ *is a* $\mathcal{D}-$*simple function then*

$$(\int \varphi dm)x = \int \varphi x dm = \int \varphi d(mx)$$

and

$$\langle(\int \varphi dm)x, z\rangle = \langle\int \varphi x dm, z\rangle = \int \varphi d\langle mx, z\rangle, \text{ for } z \in Z.$$

Proof. For the proof of parts (a), (b), and (c), see [Din00], Proposition 5.46.

To prove Assertion d), assume $S = \bigcup\limits_{n\in N} S_n$ with $S_n \in \mathcal{D}$ and consider $\varphi = 1_A$ with $A \in \mathcal{D}$. Then by the previous assertion we have

$$(\int \varphi dm)x = m(A)x = mx(A) = \int \varphi d(mx) = \int \varphi x dm.$$

By additivity we obtain the first equality of Assertion d). Replacing m with mx in the hypothesis and considering $G \subset L(G^*, \mathbb{R})$, for $z \in G^*$ we obtain the measure $(mx)z = \langle mx, z\rangle : \mathcal{D} \to \mathbb{R}$ defined by

$$\langle mx, z\rangle(A) = mx(A)z, \text{ for } a \in \mathcal{D}.$$

Applying Assertion c) we obtain

$$\langle\int \varphi d(mx), z\rangle = \int \varphi d\langle mx, z\rangle,$$

and the second equalities in Assertion d) follow. ☐

1.5 The Immediate Integral

So far we defined the integral $\int fdm$ for functions $f : S \to F$ with $f \in \mathcal{F}_{F,G(m)}$. Next we define an integral $\int fdm$ of a F-valued, $\Sigma-$measurable function $f : S \to \mathcal{F}$ with respect to the additive measure $m : \Sigma \to E \subset L(F, G)$ with bounded semivariation $\tilde{m}_{F,G}$, without assuming the $\sigma-$additivity of the measures m_z.

First, consider f to be a $\Sigma-$step function of the form $f = \sum_{i=1}^{n} \varphi_{A_i} x_i$, with $A_i \in \Sigma$ and $x_i \in F$. In that case we define $\int fdm = \sum_{i=1}^{n} m(A_i)x_i \in E$. Observe that the definition of the integral does not depend on the representation of f as a $\Sigma-$step function.

Consider now the general case. Since f is bounded and $\Sigma-$measurable, there is a sequence (f_n) of $\Sigma-$step functions such that $f_n \to f$ uniformly. Then $\lim \int f_n dm$ exists in E and we denote it by $\int fdm$. In fact,

$$\begin{aligned} |\int f_n dm - \int f_m dm| &= |\int (f_n - f_m) dm| \\ &= \|f_n - f_m\|_{\sup} \tilde{m}_{F,G}(S) \to 0, \text{ as } n, m \to \infty. \end{aligned}$$

Proposition 15. *For $x \in G$ and ϕ real-valued, bounded, $\Sigma-$measurable function we have*

$$\int \phi x dm = x \int \phi dm.$$

Proof. Since ϕ is a real-valued bounded function there is a sequence (ϕ_n) of real-valued, $\Sigma-$step functions such that $\phi_n \to \phi$ uniformly. For each n we have

$$\int \phi_n x dm = x \int \phi_n dm.$$

But $\phi_n x \to \phi x$ uniformly, hence $\int \phi_n x dm \to \int \phi x dm$. Also $x \int \phi_n dm \to x \int \phi dm$. Hence

$$\int \phi x dm = x \int \phi dm.$$

□

CHAPTER 2
ADDITIVE MEASURES

2.1 Extensions of Additive Measures

In this paragraph we present an extension theorem for additive measures in the case that $c_0 \not\subset G$.

For vector valued additive measures the canonical extension is defined differently from the canonical extension of positive measures, presented in Section 1.3.

Let $Z \subset G^*$ be a norming space for G and $m\colon \mathcal{R} \to L(F,G)$ be an additive measure, such that, for every $x \in F$ and $z \in Z$, the real-valued measure $\langle m(\cdot)x, z\rangle$ is σ-additive and has finite (resp. bounded) variation $|\langle mx, z\rangle|(\cdot)$.

Definition 16. An additive extension $m'\colon \mathcal{D} \to L(F, Z^*)$ (*respectively*, $m'\colon \Sigma \to L(F, Z^*)$) of m is called the canonical extension of m, if for every $x \in F$ and $z \in Z$, the real-valued measure $\langle m'(\cdot)x, z\rangle$ is σ-additive and has finite (*resp. bounded*) variation $|\langle m'x, z\rangle|(\cdot)$, which is the canonical extension of the variation $|\langle mx, z\rangle|(\cdot)$.

Theorem 17. *([B–D]3.7(b)) Let* $m\colon \mathcal{R} \to L(F,G)$ *be an additive measure with finite (resp. bounded) semivariation* $\tilde{m}_{F,G}$ *and* $Z \subset G^*$ *a norming space for* G^{**} *(for example,* $Z = G^*$*). Assume that for each* $z \in G^*$*, the measure* $m_z\colon \mathcal{R} \to F^*$ *is* σ*-additive. Then* m *has a canonical extension* $m'\colon \mathcal{D} \to L(F, Z^*)$ *(resp.* $m'\colon \Sigma \to L(F, Z^*)$*) with finite (resp. bounded) semivariation* $\tilde{m}'_{F,Z^*}$ *such that, for each* $z \in Z$*, the measure* $m'_z\colon \mathcal{D} \to F^*$ *(resp.* $m'_z\colon \Sigma \to F^*$*) is* σ*-additive and has finite (resp. bounded) variation* $|m'_z|$ *and we have*

$$\tilde{m}'_{F,Z^*}(A) = \tilde{m}_{F,G}(A), \quad \text{for } A \in \mathcal{R}.$$

Proof. See [B–D] 3.7 b). $\square$

Before we present the next extension theorem, we state two propositions that will be used in its proof.

Proposition 18. *([Din00]4.14(b)) If* $m : \mathcal{R} \to E = L(\mathbb{R}, E)$ *is an additive measure on a ring* $\mathcal{R}$*, then* m *is locally bounded (respectively, bounded) on* $\mathcal{R}$ *iff* $\tilde{m}_{\mathbb{R},E}(A) < \infty$ *for every* $A \in \mathcal{R}$ *(respectively,* $\tilde{m}_{\mathbb{R},E}(S) < \infty$*).*

Proposition 19. *([B–D]3.9) Let* $m : \mathcal{R} \to\subset L(F, G)$ *be an additive measure and let* $Z \subset G^*$*, be a norming space for* G*. Assume* $c_0 \not\subset E$*,* m *is locally bounded (respectively, bounded), and for every* $x \in F$ *and* $z \in Z$*, the real-valued measure* $\langle m(\cdot)x, z\rangle$ *is* $\sigma-$*additive. Then* m *is* $\sigma-$*additive and has a* $\sigma-$*additive canonical extension* $m' : \mathcal{D} \to L(F, G)$ *(respectively,* $m' : \Sigma \to L(F, G)$*).*

Theorem 20. *Assume* $c_0 \not\subset G$ *and let* $m\colon \mathcal{R} \to L(F, G)$ *be an additive measure with finite (resp. bounded) semivariation* $\widetilde{m}_{F,G}$ *and such that for each* $z \in G^*$*, the measure* $m_z\colon \mathcal{R} \to F^*$ *is* σ*-additive. Then* m *has a canonical extension* $m'\colon \mathcal{D} \to L(F, G)$ *(resp.* $m'\colon \Sigma \to L(F, G)$*) with finite (resp. bounded) semivariation* $\widetilde{m}'_{F,G}$ *such that, for each* $z \in G^*$*, the measure* $m'_z\colon \mathcal{D} \to F^*$ *(resp.* $m'_z\colon \Sigma \to F^*$*) is* σ*-additive and has finite (resp. bounded) variation* $|m'_z|$ *and we have*

$$\widetilde{m}'_{F,G}(A) = \widetilde{m}_{F,G}(A), \quad \text{for } A \in \mathcal{R}.$$

Proof. By Theorem 17 we deduce that the measure m has an unique canonical extension $m'\colon \mathcal{D} \to L(F, G^{**})$ (resp.$m'\colon \Sigma \to L(F, G^{**})$) with finite (*resp. bounded*) semivariation $\widetilde{m}'_{F,G^{**}}$ such that, for each $z \in G^*$, the measure $m'_z\colon \mathcal{D} \to F^*$ (*resp.* $m'_z\colon \Sigma \to F^*$) is σ-additive and has finite (*resp. bounded*) variation $|m'_z|$ and such that

$$\widetilde{m}'_{F,G^{**}}(A) = \widetilde{m}_{F,G}(A), \quad \text{for } A \in \mathcal{R}.$$

To prove our theorem we need to show that the extension m' is $L(F, G)-$valued rather then $L(F, G^{**})-$valued. To prove this we prove that for each $x \in F$ the measure $m'x$ is $G-$valued (rather then G^{**}-valued). In fact, since the measure m

has finite (resp. bounded) semivariation relative to (F, G) by Proposition 14 a) the measure mx has finite (resp. bounded) semivariation relative to $(\mathbb{R}, E)$. Hence by Proposition 18, the measure mx is locally bounded (resp. bounded). Moreover, by Proposition 14 a), for each $z \in G^*$ the measure $(mx)_z$ is $\sigma-$additive. Therefore we can apply Proposition 19 for the measure $mx : \mathcal{R} \to G = L(\mathbb{R}, G)$ and conclude that the measure mx is $\sigma-$additive and it has a $\sigma-$additive extension $(mx)' : \mathcal{D} \to L(\mathbb{R}, G) = G$. On the other hand the measure $m'x : \mathcal{D} \to L(\mathbb{R}, G^{**})$ is the canonical extension of the measure mx and since the canonical extension is unique we deduce that $m'x = (mx)'$ hence $m'x$ is $G-$valued, which concludes our proof. □

2.2 The Lebesgue–Stieltjes Integral for Functions with Finite Semivariation

In this section we use the same notations as in [B–D], Section 4. Namely, $I \subset \mathbb{R}$ is an interval and $g : I \to E$ is a function. We denote by $a_0 = \inf I$ and we allow it to be $-\infty$. In particular, I could be the whole $\mathbb{R}$.

Definition 21. Let $g : I \to E$ be a function. The variation of g on an interval $J \subset I$ is defined by

$$\mathrm{var}(g, J) = \sup \sum |g(t_{i+1}) - g(t_i)|,$$

where the supremum is taken over all divisions $t_0 < t_1 < \cdots < t_n$ of points from J. The function g has finite (respectively, bounded) variation if $\mathrm{var}(g, J) < \infty$ for every bounded interval $J \subset I$ (respectively, $\mathrm{var}(g, I) < \infty$).

We define the variation function $|g| : I \to [0, \infty]$ by

$$|g|(t) = \mathrm{var}(g, (a_0, t]), \text{ for } t \in I, \text{ if } a_0 \notin I,$$

and

$$|g|(t) = \mathrm{var}(g, [a_0, t]), \text{ for } t \in I, \text{ if } a_0 \in I,$$

Definition 22. Let $g\colon I \to E \subset L(F,G)$ be a function. For any interval $J \subset I$, the semivariation $\text{svar}_{F,G}(g,J)$ of g on J, relative to the embedding (F,G) is defined by

$$\text{svar}_{F,G}(g,J) = \sup \left| \sum_{i=1}^{n} [g(t_{i+1}) - g(t_i)]x_i \right|,$$

the supremum being taken for all finite divisions $t_1 < t_2 < \cdots < t_{n+1}$ of points from J and elements $x_1, x_2, \ldots, x_n$ from F with $|x_i| \leq 1$. The function g is said to have finite (resp. bounded) semivariation $\widetilde{m}_{F,G}$ if $\widetilde{m}_{F,G}(J) < \infty$ for every bounded interval $J \subset I$ (resp. $\widetilde{m}_{F,G}(I) < \infty$).

Definition 23. Let $\widetilde{g}_{F,G}\colon I \to \overline{R}_+$ be the function defined by:

$$\widetilde{g}_{F,G}(t) = \text{svar}_{F,G}(g,(a_0,t]), \text{ if } a_0 \notin I \text{ and } t \in I,$$

and

$$\widetilde{g}_{F,G}(t) = \text{svar}_{F,G}(g,[a_0,t]), \text{ if } a_0 \in I \text{ and } t \in I.$$

We call $\widetilde{g}_{F,G}$ the semivariation function of g.

For every $z \in G^*$ we define the function $g_z\colon I \to F^*$ by

$$\langle x, g_z(t) \rangle = \langle g(t)x, z \rangle, \quad \text{for } x \in F \text{ and } t \in I.$$

The next property is similar to the property in Proposition 6.

Proposition 24. *If $Z \subset G^*$ is a norming space for G, then*

$$\widetilde{g}_{F,G}(t) = \sup_{z \in Z_1} |g_z|(t), \quad \text{for } z \in Z.$$

The *jump* $\Delta g(t)$ of g at t is define by $\Delta g(t) = g(t_+) - g(t_-)$. If g is right continuous, then $\Delta g(t) = g(t) - g(t_-)$.

Proposition 25. *Assume g is right continuous.*

a) If g has finite variation $|g|$, then $\Delta|g|(t) = |\Delta g(t)|$, for $t \in \mathbb{R}$.

b) If $c_0 \not\subset E$ and g has finite semivariation $\widetilde{g}_{\mathbb{R},E}$, then $\Delta\widetilde{g}_{\mathbb{R},E}(t) = |\Delta g(t)|$.

c) We have $|\Delta g(t)| \leq \tilde{g}_{\mathbb{R},E}(t)$, *for* $t \in \mathbb{R}$.

For Assertion a) see [Din00], Theorem 18.20; for Assertion b), see [Din00], Theorem 20.14; Assertion c) follows from $|g(t) - g(s)| \leq svar(g, (-\infty, t])$, by taking the limit as $s \uparrow t$.

To each function $g : I \to E$ we associate a measure m_g. We distinguish two cases:

I) If $a_0 \notin I$, we denote by $\mathcal{P}(I)$ the semiring of the intervals $(s, t]$ with $s, t \in I$ and by $\mathcal{R}(I)$ the ring generated by $\mathcal{P}(I)$.

II) If $a_0 \in I$, we denote by $\mathcal{P}(I)$ the semiring of the intervals of the form $(s, t]$ with $s, t \in I$ and of the form $[a_0, t]$ with $a_0 < t$.

We define the measure m_g first for the intervals of the type $(s, t]$ with $a_0 < s < t$ and $[a_0, t]$, if $a_0 \in I$ and $a_0 < t$ by

$$\begin{aligned} m_g(s,t] &= g(t) - g(s), \quad \text{if } a_0 < s < t, \\ m_g[a_0,t] &= g(t) - g(a_0), \quad \text{if } a_0 < t. \end{aligned}$$

Since m_g defined above is additive, it can be extended (by additivity) to $\mathcal{R}(I)$.

Note that for every $z \in G^*$ we have $(m_g)_z = m_{g_z}$.

Theorem 26. *Assume* $c_0 \not\subset G$. *Let* $g \colon I \to E \subset L(F, G)$ *be a function with finite (respectively, bounded) semivariation* $\widetilde{g}_{F,G}$. *Assume that for every* $z \in G^*$, *the function* $g_z \colon I \to F^*$ *is right continuous on* $I \setminus \{a_0\}$. *Then the measure* m_g *has a canonical finitely additive extension* $m'_g \colon \mathcal{D}(I) \to L(F, G)$ *(resp.* $m'_g \colon \mathcal{B}(I) \to L(F, G)$*) with finite (resp. bounded) semivariation* $\widetilde{m}_{F,Z^*}$ *and such that for each* $z \in G^*$ *the measure* $(m'_g)_z \colon \mathcal{D}(I) \to F^*$ *(resp.* $(m'_g)_z \colon \mathcal{B}(I) \to F^*$*) is* σ*-additive and has finite (resp. bounded) variation.*

Proof. Note first that since the function g has finite (respectively, bounded) semivariation, the measure m_g has finite (resp. bounded) semivariation $(\widetilde{m_g})_{F,G}$ (Theorem 4.19 in [B–D]), and that for each $z \in G^*$ the function g_z has finite (resp.

bounded) variation. At the same time, by hypothesis, the function g_z is right continuous on $I \setminus \{a_0\}$. Applying Theorem 4.17 in [B-D] for the function $g_z \colon I \to F^*$ we deduce that the measure m_{g_z} can be extended uniquely to a $\sigma-$additive measure $m'_{g_z} \colon \mathcal{D}(I) \to F^*$ (resp. $m'_{g_z} : \mathcal{B}(I) \to F^*$) also with finite (resp. bounded) variation $|m'_{g_z}|$. But we remarked that $m_{g_z} = (m_g)_z$, therefore the measure $(m_g)_z : \mathcal{R}(I) \to F^*$ has a $\sigma-$additive extension to $\mathcal{D}(I)$ (resp. $\mathcal{B}(I)$). In particular for each $z \in G^*$ the measure $(m_g)_z : \mathcal{R}(I) \to F^*$ is $\sigma-$additive. Then applying Theorem 20 we conclude that the measure m_g has a canonical extension $m'_g \colon \mathcal{D}(I) \to L(F,G)$ (resp. $m'_g \colon \mathcal{B}(I) \to L(F,G)$) with finite (resp. bounded) semivariation $\widetilde{m}_{F,Z^*}$ and such that for each $z \in G^*$ the measure $(m'_g)_z \colon \mathcal{D}(I) \to F^*$ (resp. $(m'_g)_z \colon \mathcal{B}(I) \to F^*$) is σ-additive and has finite (resp. bounded) variation. $\square$

Next we define the Stieltjes integral with respect to functions $g \colon I \to E \subset L(F,G)$ with finite semivariation $\widetilde{g}_{F,G}$ and such that for every $z \in G^*$, the function $g_z \colon I \to F^*$ is right continuous on $I \setminus \{a_0\}$. In case $c_0 \not\subset G$ we can apply the previous theorem and deduce that the measure m_g has an additive extension to the whole Borel $\sigma-$algebra $\mathcal{B}(I)$ and such that for every $z \in G^*$ the extended measure $(m_g)_z$ is $\sigma-$additive. In this case we can apply the integration theory developed in section 1.4 and define the space $\mathcal{F}_{F,G}(m_g)$ to be the space of all m_g-measurable functions $f : I \to F$ such that

$$\tilde{m}_{F,G}(f) = \sup\{\int |f| d|m_g|_z : z \in G_1^*\} < \infty,$$

and then the integral $\int f dm_g$ to be the continuous, linear functional $z \mapsto \int f d(m_g)_z$ on G^*, hence belonging to G^{**}.

We denote $\mathcal{F}_{F,G}(m_g)$ by $\mathcal{F}_{F,G}(g)$ and define the Lebesgue-Stieltjes integral $\int f dg$ by the equality

$$\int f dg = \int f dm_g.$$

2.3 Optional and Predictable Measures

In this section we present several notations that will be used in the rest of this work, as well as the definition and properties of optional and predictable, finitely additive measures.

- $(\Omega, \mathcal{F}, P)$ is a *probability space.* $P-$negligible sets would be called, simply, negligible sets. A set $M \subset \mathbb{R}_+ \times \Omega$ is called *evanescent* if it is contained in a set of the form $\mathbb{R}_+ \times A$ with $A \subset \Omega$, negligible.
- $(\mathcal{F}_t)_{t \in \mathcal{R}_+}$ is a filtration satisfying the *usual conditions.*, i.e., $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for every $t \geq 0$ and $\mathcal{F}_0$ contains all negligible sets.
- A stopping time is a function $T : \Omega \rightarrow \overline{R}_+$ such that $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. If $S \leq T$ are two stopping times we define a stochastic interval $(S, T]$ by the equality

$$(S, T] = \{t, \omega) \in R_+ \times \Omega : S(\omega) < t < T(\omega)\}.$$

The stochastic intervals $[S, T]$, $[S, T)$, (S, T) are defined similarly.

- A function $X : \mathbb{R}_+ \times \Omega \rightarrow E$ is called process. We use the notation $X(t, \omega) = X_t(\omega)$. A process X is called adapted if X_t is $\mathcal{F}_t-$measurable for every $t \geq 0$.
- Let $\mathcal{R}$ be the ring of subsets of $\mathbb{R}_+ \times \Omega$, generated by the semiring of predictable rectangles of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$ and $(s, t] \times A$ with $A \in \mathcal{F}_s$. The σ-algebra generated by $\mathcal{R}$ is called the *predictable* σ-algebra and is denoted by $\mathcal{P}$. The predictable σ-algebra is also generated by the adapted, left continuous, real-valued processes. It is also generated by the stochastic intervals $(S, T]$ with $S \leq T$ stopping times.

A $\mathcal{P}$-measurable process is called a *predictable* process.

A stopping time T is said to be *predictable* if the stochastic interval $[T, \infty)$ is predictable.

The *optional* σ-algebra $\mathcal{O}$ is the σ-algebra generated by the adapted, right continuous real valued processes. It is also generated by the stochastic intervals $[T, \infty)$

with $S \leq T$ simple stopping times. An $\mathcal{O}$-measurable process is called an *optional process.*

- $\mathcal{B}(\mathcal{R}_+) \times \mathcal{F}$ is denoted by $\mathcal{M}$. Processes measurable with respect to $\mathcal{M}$ will be called measurable processes.
- $X : \mathbb{R}_+ \times \Omega \to E$ is a *cadlag* (i.e., right continuous, with left limits), *adapted* process, with $X_t \in L^p_E$ for every $t \geq 0$. Such a process has separable range.

We consider X automatically extended on $\mathbb{R} \times \Omega$, with $X_t = 0$ for $t < 0$. Then $X_{0-} = 0$. We extend also the filtration with $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$.

- We use the notations L^p_M for $L^p_M(P)$.
- Remark that if $E \subset L(E, F)$, then $L^p_E \subset L(F, L^p_G)$.

Definition 27. Let φ be a real-valued, bounded, measurable process. We denote by ${}^o\varphi$ (resp. ${}^p\varphi$) the optional (resp. predictable) dual projection of φ, defined by

$$(\varphi_T I_{T<\infty} | \mathcal{F}_T) = {}^o\varphi_T I_{T<\infty}, a.s.,$$

for every stopping time T; (resp.

$$(\varphi_T I_{T<\infty} | \mathcal{F}_T) = {}^p\varphi_T I_{T<\infty}, a.s.,$$

for every stopping time T.)

Let $m : \mathcal{M} \to E \subset L(F, G)$ be an additive measure, with bounded semivariation $\tilde{m}_{F,G}$ relative to (F, G), such that for each $z \in G^*$ the measure $m_z : \mathcal{M} \to F^*$ is $\sigma-$additive.

Definition 28. We say the measure m is optional (resp. predictable) relative to (F, G) if for every set $A \in \mathcal{M}$ we have

$$m(A) = \int {}^o(1_A) dm$$

(resp.

$$m(A) = \int {}^p(1_A) dm).$$

If the pair (F, G) is not specified then we assume it is the pair $(\mathbb{R}, E)$ and we call the measure simply optional (resp. predictable).

Remark: Since the functions ${}^o(1_A)$ and ${}^p(1_A)$ are bounded the above integrals are defined as the integrals in Section 1.4.

Proposition 29. *m is optional (resp. predictable) relative to (F, G) iff for every real-valued, bounded, measurable process ϕ and for every $x \in F$ we have*

$$\int \phi dm = \int {}^o\phi dm, \tag{*}$$

(resp.

$$\int \phi dm = \int {}^p\phi dm.)$$

Proof. Note first that since ϕ, ${}^o\phi$ and ${}^p\phi$ are bounded all the integrals are defined as in Section 1.4.

Assume first the equality (*) is true. If we take $\phi = 1_A \in \mathcal{M}$ we obtain

$$m(A) = \int 1_A dm = \int {}^o(1_A) dm.$$

Hence the measure m is optional relative to (F, G).

Then assume that m is optional relative to (F, G) and prove the equality (*). Let ϕ be a real-valued, bounded, measurable process. Since m is optional the equality (*) is true for $\mathcal{M}-$step processes. Since ϕ is a real-valued, bounded, measurable process there is a sequence of $\mathcal{M}-$step processes ϕ^n such that $|\phi^n| \leq |\phi|$ and $\phi^n \to \phi$ uniformly. Then

$$\int \phi^n dm \to \int \phi dm.$$

Also since ${}^o\phi^n \to {}^o\phi$ uniformly we have

$$\int {}^o\phi^n dm \to \int {}^o\phi dm.$$

Because $\int \phi^n dm_z = \int {}^o\phi^n dm_z$ (the equality (*) is true for $\mathcal{M}-$step processes) we infer that

$$\int \phi dm = \int {}^o\phi dm.$$

In the predictable case the proof is similar. □

2.4 Dual Projections of Measures

In this section we present the definition and properties of the optional and predictable dual projection of a measure m.

Let $m : \mathcal{M} \to E \subset L(F, G)$ be an additive measure with bounded semivariation $\tilde{m}_{F,G}$, relative to (F, G), such that for each $z \in G^*$ the measure m_z is $\sigma-$additive.

Let A be a set in $\mathcal{M}$ and consider the optional projection ${}^o(1_A)$ and the predictable projection ${}^p(1_A)$ of 1_A. Define the measures $m^o, m^p : \mathcal{M} \to G$ by

$$m^o(A) = \int {}^o(1_A)dm,$$

and

$$m^p(A) = \int {}^p(1_A)dm.$$

Definition 30. The measures $m^o, m^p : \mathcal{M} \to E \subset L(F, G)$ are called the optional respectively the predictable dual projection of m.

Theorem 31. *The measures $m^o, m^p : \mathcal{M} \to E \subset L)F, G)$ are additive and for each $z \in G^*$ the measures $m^o_z, m^p_z : \mathcal{M} \to F^*$ are $\sigma-$additive. Moreover both measures m^o and m^p have finite semivariation relative to (F, G) . If m has finite variation then so do m^o and m^p, and we have*

$$|m^o| \leq |m|^o, \text{ and}$$
$$|m^p| \leq |m|^p.$$

Proof. The fact that m^o and m^p are additive is obvious. To prove that m_z^o and m_z^p are $\sigma-$additive for each $z \in G^*$, we first prove that for each $z \in G^*$ we have

$$(m^o)_z = (m_z)^o, \text{ and } (m^p)_z = (m_z)^p.$$

In fact, let $A \in \mathcal{M}$ and $x \in F$. Then by Proposition 5.46 d) in [Din00] we have

$$\begin{aligned}\langle x, (m^o)_z(A)\rangle &= \langle m^o(A)x, z\rangle = \langle \int {}^o(1_A)xdm, z\rangle \\ &= \int {}^o(1_A)xdm_z = x\int {}^o(1_A)dm_z \\ &= x(m_z)^o(A) = \langle x, (m_z)^o(A)\rangle,\end{aligned}$$

Therefore we obtain $(m^o)_z = (m_z)^o$. Since m_z is $\sigma-$additive we deduce by Theorem 22.1 in [Din00] that the measure $(m_z)^o$ is $\sigma-$additive, hence the measure $(m^o)_z$ is $\sigma-$additive also.

Next we prove that the measure m^o has finite semivariation. This is a direct consequence of the previous equality and of Proposition 6. By Proposition 6 we deduce that for each $z \in G^*$ the measure m_z has finite variation, hence, by Theorem 22.1 in [Din00], the measure $(m_z)^o$ has finite variation, which implies the fact that the measure $(m^o)_z$ has finite variation. Since this is true for each $z \in G^*$ we conclude that the measure m^o has finite semivariation relative to (F, G).

Assume next that the measure m has finite variation. By the definition of m^o we have

$$|m^o(A)| = \int {}^o(1_A)d|m| \leq |m|(A) < \infty,$$

which concludes our proof.

The predictable case is proved similarly.

$\square$

Theorem 32. *Let* $m : \mathcal{M} \to E \subset L(F,G)$ *be an additive measure with bounded semivariation* $\tilde{m}_{F,G}$, *relative to* (F,G), *such that for each* $z \in G^*$ *the measure* m_z *is* $\sigma-$*additive. Then for every real valued, measurable, bounded process* ϕ *we have*

$$\int \phi dm^o = \int {}^o\phi dm \text{ and } \qquad (*)$$
$$\int \phi dm^p = \int {}^p\phi dm$$

Proof. Let $\phi = 1_A$ with $A \in \mathcal{M}$. Then the equality (*) is equivalent to

$$m^o(A) = \int {}^o(1_A)dm.$$

But by the definition of m^o we have

$$m^o(A) = \int {}^o(1_A)dm.$$

hence the equality (*) is verified. By additivity, we deduce that the equality (*) is verified for every $\mathcal{M}-$step, real-valued process ϕ.

Consider now an arbitrary real-valued, measurable, bounded process ϕ. Then there is a sequence ϕ^n of $\mathcal{M}-$step process such that $\phi^n \to \phi$ uniformly and $|\phi^n| \le |\phi|$. Then

$$\int \phi^n dm^o \to \int \phi dm^o.$$

Moreover we have ${}^o\phi^n \to {}^o\phi$ uniformly. Hence, by the uniform convergence theorem,

$$\int {}^o\phi^n dm = \int {}^o\phi dm.$$

Since for each n we have $\int {}^o\phi^n dm = \int \phi^n dm^o$, we deduce that

$$\int {}^o\phi dm = \int \phi dm^o,$$

which is equivalent to equality (*).

The predictable case is treated similarly. $\square$

CHAPTER 3
ADDITIVE SUMMABLE PROCESSES

In this chapter we present the definition of additive summable processes, and we define the stochastic integral with respect to such a process.

3.1 The Measure I_X

In this section we shall first define the measure I_X.

Let $X : \mathbb{R}_+ \times \Omega \to E$ be a cadlag, adapted process with $X_t \in L^p_E$ for every $t \geq 0$ and $1 \leq p < \infty$.

- Let $\mathcal{S}$ be the semiring of predictable rectangles and $I_X : \mathcal{S} \to L^p_E$ the stochastic measure defined by

$$I_X(\{0\} \times A) = 1_A X_0, \text{ for } A \in \mathcal{F}_0$$

and

$$I_X((s,t] \times A) = 1_A(X_t - X_s), \text{ for } A \in \mathcal{F}_s.$$

Note that I_X is finitely additive on $\mathcal{S}$ therefore it can be extended uniquely to a finitely additive measure on the ring $\mathcal{R}$ generated by $\mathcal{S}$. We obtain a finitely additive measure $I_X : \mathcal{R} \to L^p_E$ verifying the previous equalities. Besides, we have

$$I_X([0,t] \times A) = 1_A X_t, \text{ for } t \geq 0 \text{ and } A \in \mathcal{F}_0.$$

In particular, for $A = \Omega$ we have

$$I_X([0,t] \times \Omega) = X_t, \text{ for } t \geq 0.$$

- Since $L^p_E \subset L(F, L^p_G)$, we can consider the semivariation of I_X relative to the pair (F, L^p_G). To simplify the notation, we shall write I instead of I_X and $\tilde{I}_{F,G}$ or $(\tilde{I}_X)_{F,G}$

instead of $(\tilde{I}_X)_{F,L^p_G}$ and we shall call it the semivariation of I relative to (F,G):

$$\tilde{I}_{F,G}(A) = \sup \| \sum I_X(A_i)x_i \|_p, \text{ for } A \in \mathcal{R},$$

where the supremum is taken for all finite families $(A_i)_{i \in I}$ of mutually disjoint sets from $\mathcal{R}$ contained in A and all families $(x_i)_{i \in I}$ of elements from F_1.

3.2 The Measures $(I_X)_z$

In this section we define the measures $(I_X)_z$.

Let $X : \mathbb{R}_+ \times \Omega \to E$ be a cadlag, adapted process with $X_t \in L^p_E$ for every $t \geq 0$.

Consider the additive measure $I_X : \mathcal{R} \to L^p_E \subset L(F, L^p_G)$ defined above.

Let $Z \subset (L^p_G)^*$ be a norming space for L^p_G. We denote $m = I_X$ and for any $z \in Z$ we consider the measure

$$m_z = (I_X)_z : \mathcal{R} \to F^*$$

defined for $A \in \mathcal{P}$ and $y \in F$ by

$$\langle y, m_z(A) \rangle = \langle m(A)y, z \rangle = \int \langle I_X(A)(\omega)y, z(\omega) \rangle dP(\omega),$$

where the bracket in the integral represents the duality between G and G^*.

3.3 Additive Summable Processes

In this section we give a definition of summable process and of additive summable processes, definition that is weaker than the definition of summable processes. We use these processes to define the stochastic integral in Section 4.

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be a cadlag, adapted process, with $X_t \in L^p_E$ for $t \geq 0$.

Definition 33. We say that X is p-summable relative to the pair (F,G) if I_X has an $\sigma-$additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to (F,G).

Definition 34. We say that X is p-additive summable relative to the pair (F,G) if I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to (F,G) and the measure $(I_X)_z$ is σ-additive for each $z \in (L^p_G)^*$.

If $p = 1$, we say, simply, that X is additive summable relative to (F,G).

Remark: Note that in the paper "The Riesz representation theorem and extension of vector valued additive measures" N. Dinculeanu and B. Bongiorno [BD01] (Theorem 3.7 II) proved that if each of the measures $(I_X)_z$ is σ-additive and if $I_X : \mathcal{R} \to L^p_E$ has finite semivariation relative to (F,G) then I_X has canonical additive extension $I_X : \mathcal{P} \to (L^p_E)^{**}$ with finite semivariation relative to $(F,(L^p_E)^{**})$ such that for each $z \in (L^p_G)^*$, the measure $(I_X)_z$ is $\sigma-$additive on $\mathcal{P}$ and has finite variation $|(I_X)_z|$.

Proposition 35. *If X is $p-$additive summable relative to* $(\mathbb{R},E)$ *then X is $p-$summable relative to* $(\mathbb{R},E)$.

Proof. If X is $p-$additive summable relative to $(\mathbb{R},E)$ then the measure I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to $(\mathbb{R},E)$. Moreover for each $z \in (L^p_E)^*$ the measure $(I_X)_z$ is σ-additive. By Pettis Theorem, the measure I_X is $\sigma-$additive. Hence, the process X is $p-$summable. ☐

3.4 The Space $\mathcal{F}_D(\tilde{I}_{F,L^p_G})$

In this section we define the set of processes integrable with respect I_X.

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be a cadlag, adapted process and $Z = (L^p_G)^*$. Assume X is p-additive summable relative to (F,G).

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F,L^p_G)$ with bounded semivariation $\tilde{I}_{F,G}$ relative to (F,L^p_G) for which each measure $(I_X)_z$ is σ-additive for every $z \in Z$.

Then we have

$$(\tilde{I}_X)_{F,L^p_G} = \tilde{m}_{F,L^p_G} = \sup\{|m_z| : z \in Z, \|z\| \leq 1, z \in (L^p_F)^*\}.$$

If p is fixed, to simplify the notation, we can write $\tilde{I}_{F,G} = \tilde{I}_{F,L^p_G}$.

For any Banach space D we denote by $\mathcal{F}_D(\tilde{I}_{F,G})$ or $\mathcal{F}_D(\tilde{I}_{F,L^p_G})$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to D$ such that

$$\tilde{I}_{F,G}(H) = \tilde{m}_{F,L^p_G}(H) = \sup\{\int |H|d|m_z| : \|z\|_q \leq 1\} < \infty.$$

3.5 The Integral $\int HdI_X$

In this section we define the integral $\int HdI_X$.

- If $D = F$ we can define the integral $\int HdI_X \in (L^p_G)^{**}$ for any $H \in \mathcal{F}_F(\tilde{I}_{F,G})$. To simplify the notation we write $\mathcal{F}_{F,G}(X)$ instead of $\mathcal{F}_F((\tilde{I}_X)_{F,G})$ or $\mathcal{F}_F((\tilde{I}_X)_{F,L^p_G})$. Let $H \in \mathcal{F}_{F,G}(X)$; then

$H \in L^1_F(|(I_X)_z|)$ for every $z \in (L^p_G)^*$, hence the integral $\int Hd(I_X)_z$ is defined and is a scalar. The mapping $z \mapsto \int Hd(I_X)_z$ is a continuous linear functional on $(L^p_G)^*$, denoted $\int HdI_X$. Therefore, $\int HdI_X \in (L^p_G)^{**}$,

$$\langle \int HdI_X, z\rangle = \int Hd(I_X)_z, \text{ for } z \in Z$$

and

$$|\int HdI_X| \leq \tilde{I}_{F,G}(H).$$

3.6 Lebesgue Convergence for the Integral $\int HdI_X$

In this section we state a convergence theorem of Lebesgue type, for the pointwise convergence $H^n \to H$. We obtain a weak convergence in L^p_G for the integrals.

Theorem 36. *Let* $(H^n)_{0\leq n<\infty}$ *be a sequence of elements from* $\mathcal{F}_{F,G}(X)$ *such that* $|H^n| \leq |H^0|$ *for each* n *and* $H^n \to H$ *pointwise. Assume that*

(i) $\int H^n dI_X \in L^p_G$ *for every* $n \geq 1$

and

(ii) *The sequence* $(\int H^n dI_X)_n$ *converges pointwise on* Ω*, weakly in* G.

Then

a) $\int H dI_X \in L^p_G$

and

b) $\int H^n dI_X \to \int H dI_X$, *in the weak topology of* L^p_G, *as well as pointwise, weakly in* G.

c) *If* $(\int H^n dI_X)_n$ *converges pointwise on* Ω, *strongly in* G, *then*

$$\int H^n dI_X \to \int H dI_X,$$

strongly in L^1_G.

Proof. This theorem was proved in [Din00] under the assumtion that I_X is $\sigma-$additive. The same proof remains valid in our case, where I_X is only finitely additive. □

CHAPTER 4
THE STOCHASTIC INTEGRAL

In this chapter we define the stochastic integral (Section 1) and we prove some properties of the stochastic integral.

4.1 Definition of the Stochastic Integral $H \cdot X$

In this section we define the stochastic integral and we prove that the stochastic integral is a cadlag adapted process.

Let $H \in \mathcal{F}_{F,G}(X)$. Then, for every $t \geq 0$ we have $1_{[0,t]}H \in \mathcal{F}_{F,G}(X)$. We denote by $\int_{[0,t]} H dI_X$ the integral $\int 1_{[0,t]} H dI_X$. We define

$$\int_{[0,\infty]} H dI_X := \int_{[0,\infty)} H dI_X = \int H dI_X.$$

Taking $Z = (L^p_G)^*$, for each $H \in \mathcal{F}_{F,G}(X)$ we obtain a family $(\int_{[0,t]} H dI_X)_{t \in \mathbb{R}_+}$ of elements of $(L^p_G)^{**}$.

We restrict ourselves to processes H for which $\int_{[0,t]} H dI_X \in L^p_G$ for each $t \geq 0$. Since L^p_G is a set of equivalence classes, $\int_{[0,t]} H dI_X$ represents an equivalence class. We use the same notation for any random variable in its equivalence class. We are interested to see whether or not the process $(\int_{[0,t]} H dI_X)_{t \geq 0}$ is adapted and if it admits a cadlag modification.

The next theorem was proved in [Din00] in case I_X is $\sigma-$additive. The same proof is valid in our case, where I_X is only additive. For the sake of completeness, we reproduce the proof below.

Theorem 37. *Let $X : \mathbb{R} \to E \subset L(F, G)$ be a cadlag, adapted, p-summable process relative to (F, G) and $H \in \mathcal{F}_{F,G}(X)$ such that $\int_{[0,t]} H dI_X \in L^p_G$ for every $t \geq 0$.*

Then the process $(\int_{[0,t]} H dI_X)_{t \geq 0}$ is adapted.

Proof. The process $(\int_{[0,t]} HdI_X)_{t\geq 0}$ is adapted iff for every $t \geq 0$ the random variable $\int_{[0,t]} HdI_X$ is $\mathcal{F}_t$-measurable. Let $t \geq 0$ Assume first that $H = 1_M x$ with $M \in \mathcal{R}$ and $x \in F$. If $M = \{0\} \times A$ with $A \in \mathcal{F}_0$, then

$$\int_{[0,t]} HdI_X = \int 1_{\{0\}\times A} x dI_X = I_X(\{0\} \times A\}x = 1_A X_0 x$$

and $1_A X_0 x$ is $\mathcal{F}_0$-measurable.

Assume $M = (u, v] \times A$ with $A \in \mathcal{F}_u$; if $t \leq u$, then

$$\int_{[0,t]} HdI_X = 0;$$

if $u < t$, then $([0,t] \times \Omega) \cap ((u,v] \times A) = (u, t \wedge v] \times A$ and

$$\int_{[0,t]} HdI_X = 1_A(X_{t\wedge v} - X_u)x,$$

hence we deduce that $\int_{[0,t]} HdI_X$ is $\mathcal{F}_t-$measurable. We deduce then that $\int_{[0,t]} HdI_X$ is $\mathcal{F}_t$-measurable if $H = 1_M x$ with $M \in \mathcal{R}$, since any $M \in \mathcal{R}$ can be written as a finite union of disjoint sets from the semiring of predictable rectangles and I_X is a finitely additive measure.

Next we prove that $\int_{[0,t]} HdI_X$ is $\mathcal{F}_t-$measurable if $H = 1_M x$ with $M \in \mathcal{P}$ and $x \in F$. In fact, denote by $\mathcal{P}_0$ the class of the sets $M \in \mathcal{P}$ such that $\int_{[0,t]} 1_M x dI_X$ is $\mathcal{F}_t$-measurable for every $x \in F$. By the previous argument $\mathcal{P}_0$ contains $\mathcal{R}$. We prove now that $\mathcal{P}_0$ is a monotone class. Let (M_n) be a monotone sequence from $\mathcal{P}_0$ with limit M and prove that $M \in \mathcal{P}_0$ and let $x \in F$.

Let $z \in (L^p_G)^*$. Since $|1_{[0,t]}1_{M_n}x| \to |1_{[0,t]}1_M x|$, by the Lebesgue dominant convergence theorem we have $1_{[0,t]}1_{M_n}x \to 1_{[0,t]}1_M x$ in $L^1_F((I_X)_z)$; therefore

$$\int_{[0,t]} 1_{M_n} x d(I_X)_z \to \int_{[0,t]} 1_M x d(I_X)_z,$$

which is equivalent to,

$$\langle \int_{[0,t]} 1_{M_n} x dI_X, z\rangle \to \langle \int_{[0,t]} 1_M x dI_X, z\rangle.$$

Therefore

$$\int_{[0,t]} 1_{M_n} x dI_X \to \int_{[0,t]} 1_M x dI_X$$

weakly in $L^p_G = L^p_G(\mathcal{F}, P)$. Since for each n the random variable $\int_{[0,t]} 1_{M_n} x dI_X$ is $\mathcal{F}_t$-measurable and is an element of L^p_G, we deduce that it is an element of $L^p_G(\mathcal{F}_t, P)$. Therefore the weak limit $\int_{[0,t]} 1_M x dI_X$ is an element of the weak closure of $L^p_G(\mathcal{F}_t, P)$ in the space $L^p_G(\mathcal{F}, P)$. Since $L^p_G(\mathcal{F}_t, P)$ is a convex set the weak closure is equal to the strong closure in the space $L^p_G(\mathcal{F}_t, P)$. But the strong closure of $L^p_G(\mathcal{F}_t, P)$ in $L^p_G(\mathcal{F}, P)$ is equal to $L^p_G(\mathcal{F}_t, P)$. Therefore $\int_{[0,t]} 1_M x dI_X$ is $\mathcal{F}_t-$measurable. We conclude that $\mathcal{P}_0$ is a monotone class; consequently, $\mathcal{P}_0 = \mathcal{P}$. Hence we proved that $\int_{[0,t]} H dI_X$ is $\mathcal{F}_t$-measurable for every $\mathcal{P}$-step process $H : \mathbb{R}_+ \times \Omega \to F$.

Now we are looking at the general case, with H a predictable process in $\mathcal{F}_{F,G}(X)$ such that $\int_{[0,t]} H dI_X \in L^p_G$ for each $t \geq 0$. Since H is predictable, there is a sequence (H^n) of predictable step processes such that $H^n \to H$ pointwise and $|H^n| \leq |H|$ everywhere, for every n.

Let $z \in (L^p_G)^*$. Then we have $H \in L^1_F((I_X)_z)$. We can apply Lebesgue's Theorem in this space, and deduce that $1_{[0,t]} H^n \to 1_{[0,t]} H$ in $L^1_F((I_X)_z)$. Hence

$$\int_{[0,t]} H^n d(I_X)_z \to \int_{[0,t]} H d(I_X)_z,$$

or,

$$\langle \int_{[0,t]} H^n dI_X, z\rangle \to \langle \int_{[0,t]} H dI_X, z\rangle.$$

We deduce that

$$\int_{[0,t]} H^n dI_X \to \int_{[0,t]} H dI_X \text{ weakly in } L^p_F(\mathcal{F}, P).$$

Since H^n are simple we have $\int_{[0,t]} H^n dI_X \in L_F^p(\mathcal{F}_t, P)$ for every n, and using the same argument as before we deduce that $\int_{[0,t]} H dI_X \in L_F^p(\mathcal{F}_t, P)$ that is, $\int_{[0,t]} H dI_X$ is $\mathcal{F}_t$-measurable $\square$

It is not clear that there is a cadlag modification of the previously defined process $(\int_{[0,t]} H dI_X)_t$. Therefore we use the following definition

Definition 38. We define $L^1_{F,G}(X)$ to be the set of processes $H \in \mathcal{F}_{F,G}(I_X)$ that satisfy the following two conditions:

a) $\int_{[0,t]} H dI_X \in L_G^p$ for every $t \in \mathbb{R}_+$;

b) The process $(\int_{[0,t]} H dI_X)_{t \geq 0}$ has a cadlag modification.

The processes $H \in L^1_{F,G}(X)$ are said to be integrable with respect to X.

If $H \in L^1_{F,G}(X)$, then any cadlag modification of the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ is called the stochastic integral of H with respect to X and is denoted by $H \cdot X$ or $\int H dX$:

$$(H \cdot X)_t(\omega) = (\int H dX)_t(\omega) = (\int_{[0,t]} H dI_X)(\omega), \text{ a.s.}$$

Therefore the stochastic integral is defined up to an evanescent process. For $t = \infty$ we have

$$(H \cdot X)_\infty = \int_{[0,\infty]} H dI_X = \int_{[0,\infty)} H dI_X = \int H dI_X.$$

Note that if $H : \mathbb{R}_+ \times \Omega \to F$ is an $\mathcal{R}$-step process then we have

$$(H \cdot X)_t(\omega) = \int_{[0,t]} H_s(\omega) dX_s(\omega),$$

that is, the stochastic integral can be computed pathwise.

From the previous theorem and definition we deduce the following theorem:

Theorem 39. *If $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ is a p-additive summable process relative to (F,G) and if $H \in L^1_{F,G}(X)$, then the stochastic integral $H \cdot X$ is a cadlag, adapted process.*

The next theorem shows that the stochastic integral $H \cdot X$ is cadlag in L_G^p.

Theorem 40. *Let $H \in L^1_{F,G}(X)$.*

a) *For every $t \in [0,\infty)$ we have $(H \cdot X)_{t-} \in L^p_G$ and*

$$(H \cdot X)_{t-} = \int_{[0,t)} H dI_X, \ a.s.$$

If $(H \cdot X)_{\infty-}(\omega)$ exists for each $\omega \in \Omega$, then

$$(H \cdot X)_{\infty-} = (H \cdot X)_{\infty} = \int H dI_X, \ \ a.s.$$

b) *The mapping $t \mapsto (H \cdot X)_t$ is cadlag in L^1_G.*

Proof. Let $t \geq 0$ and let $t_n \uparrow t$. If $(H \cdot X)_{\infty-}$ exists, we allow $t = \infty$. Since the process $H \cdot X$ is cadlag we deduce that $(H \cdot X)_{t_n} \to (H \cdot X)_{t-}$ pointwise on Ω, strongly in G. On the other hand we have $1_{[0,t_n]}H \to 1_{[0,t)}H$ pointwise and $|1_{[0,t_n]}H| \leq |1_{[0,t)}H| \leq |H|$ for every n. Also $\int 1_{[0,t_n]} H dI_X = (H \cdot X)_{t_n} \in L^p_G$. Applying Theorem 36 we obtain $\int 1_{[0,t)} H dI_X \in L^p_G$ and

$$\int 1_{[0,t_n]} H dI_X \to \int 1_{[0,t)} H dI_X, \ \text{in } L^1_G.$$

Therefore $(H \cdot X)_{t-} = \int_{[0,t)} H dI_X$ a.s, hence $(H \cdot X)_{t-} \in L^p_G$. This proves Assertion a).To prove Assertion b) observe that the mapping $t \mapsto (H \cdot X)_t$ from $\overline{\mathbb{R}}_+$ to L^1_G has left limit at every $t < \infty$. Notice that it has an left hand side limit at $t = \infty$, if $(H \cdot X)_{\infty-}(\omega)$ exists for every $\omega \in \Omega$. Taking $t_n \downarrow t$, a similar proof shows that $(H \cdot X)_{t_n} \to (H \cdot X)_t$ in L^1_G, hence the mapping $t \mapsto (H \cdot X)_t$ is cadlag in L^1_G. $\square$

4.2 The Stochastic Integral and Stopping Times

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be an additive summable process

Proposition 41. *For any stopping time T we have $X_T \in L^p_E$ and $I_X[0,T] = X_T$ for T simple. For any decreasing sequence (T_n) of simple stopping times such that $T_n \downarrow T$, and for every $z \in (L^p_G)^*$ we have*

$$\langle I_X([0,T])y, z\rangle = \lim_n \langle X_{T_n} y, z\rangle, \tag{4.1}$$

where the bracket represents the duality between L^p_G *and* $(L^p_G)^*$.

Proof. Assume first that T is a simple stopping time of the form

$$T = \sum_{1 \leq i \leq n} 1_{A_i} t_i,$$

with $0 < t_i \leq \infty$, $t_i \neq t_j$ for $i \neq j$, $A_i \in \mathcal{F}_{t_i}$ are mutually disjoint and $\bigcup_{1 \leq i \leq n} A_i = \Omega$. Then $[0, T] = \bigcup_{1 \leq i \leq n} [0, t_i] \times A_i$ is a disjoint union. Hence $I_X([0,T]) = \sum_i I_X([0, t_i] \times A_i) = \sum_i 1_{A_i} X_{t_i} = X_T$. Since $I_X : \mathcal{P} \to L^p_E$, we conclude that $X_T \in L^p_E$.

Next, assume that (T_n) is a sequence of simple stopping times such that $T_n \downarrow T$. Then $[0, T_n] \downarrow [0, T]$. Since $(I_X)_z$ is σ-additive in F^*, for any $y \in F$ and $z \in (L^p_G)^*$, we have

$$\begin{aligned} \langle I_X([0,T])y, z\rangle &= \langle (I_X)_z([0,T]), y\rangle \\ &= \lim_{n\to\infty} \langle (I_X)_z([0,T_n]), y\rangle \\ &= \lim_{n\to\infty} \langle I_X([0,T_n]y, z\rangle \\ &= \lim_{n\to\infty} \langle X_{T_n} y, z\rangle. \end{aligned}$$

and the relation (4.1) is proven. It remains to prove that $X_T \in L^p_E$. Since $X_{T_n}(\omega) \to X_T(\omega)$ it follows that X_T is $\mathcal{F}-$measurable. We will prove that $|X_{T_n}| \in L^p$ to deduce that $X_{T_n} \in L^p_G$.

We saw before that for $y \in F$ and $z \in (L^p_G)^*$ the sequence $\langle (I_X)([0,T_n])y, z\rangle$ is convergent hence bounded, i.e.

$$\sup_n |\langle (I_X)([0,T_n])y, z\rangle| < \infty, \text{ for } y \in F, z \in (L^p_G)^*.$$

By the Banach-Steinhauss Theorem, we have

$$\sup_n \|I_X([0,T_n]y\|_{L^p_G} < \infty, \text{ for } y \in F$$

hence

$$\sup_n \|I_X([0,T_n]\|_{L^p_E} < \infty.$$

or $\sup_n \|X_{T_n}]\|_{L^p_E} < \infty$, which is equivalent to $\sup_n \int |X_{T_n}|^p dP < \infty$. Now $|X_T|^p = \lim |X_{T_n}|^p = \liminf |X_{T_n}|^p$. If we apply Fatou Lemma we get:

$$\int |X_T|^p dP = \int \liminf |X_{T_n}|^p \leq \liminf \int |X_{T_n}|^p dP \leq \sup \int |X_{T_n}|^p dP < \infty.$$

therefore $X_T \in L^p_G$. ☐

Proposition 42. *Let $S \leq T$ be stopping times and $h : \Omega \to F$ be an $\mathcal{F}_S$-measurable, simple random variable. Then for any pair $(T^n)_n$, $(S^n)_n$ of sequences of simple stopping times, with $T^n \downarrow T, S^n \downarrow S$, such that $S^n \leq T^n$ for each n, we have*

$$\langle \int h1_{(S,T]}\, dI_X, z\rangle = \lim_n \langle h(X_{T^n} - X_{S^n}), z\rangle, \text{ for } z \in (L^p_G)^*, \tag{4.2}$$

where the bracket represents the duality between L^p_G and $(L^p_G)^*$.

Proof. First we prove that there are two sequences (T^n) and (S^n) of simple stopping times such that $T^n \downarrow T, S^n \downarrow S$ and $S^n \leq T^n$. In fact, there are two sequences of simple stopping times T^n and P^n such that $P^n \downarrow S$ and $T^n \downarrow T$. Consider, now, $S^n = P^n \wedge T^n$. Since P^n and T^n are stopping times, S^n is a stopping time and $S^n \leq T^n$. On the other hand, observe that $S \leq S^n \leq P^n$ and $\lim P^n = S$. Therefore $lim_{n\to\infty} S^n = S$ too. So we have $S^n \downarrow S$ and $S^n \leq T^n$.

Now we want to prove (4.2). Assume first $h = 1_A y$ with $A \in \mathcal{F}_S$ and $y \in F$. Then

$$\begin{aligned}\int h1_{(S,T]}\, dI_X &= \int 1_A y 1_{(S,T]}\, dI_X \\ &= \int 1_{(S_A,T_A]} y\, dI_X \\ &= I_X((S_A,T_A])y.\end{aligned}$$

For any sequence of simple stopping times (T^n) and (S^n) with $T^n \downarrow T$, $S^n \downarrow S$ and $S^n \leq T^n$, we have $T^n_A \downarrow T_A$ and $S^n_A \downarrow S_A$. Therefore, applying Proposition 41 for every $z \in (L^p_G)^*$, we get

$$\begin{aligned}
\langle \int h1_{(S,T]}\, dI_X, z\rangle &= \langle I_X((S_A, T_A])y, z\rangle \\
&= \langle [I_X([0, T_A]) - I_X([0, S_A])]y, z\rangle \\
&= \lim_{n\to\infty} \langle X_{T^n_A} y, z\rangle - \lim_{n\to\infty} \langle X_{S^n_A} y, z\rangle \\
&= \lim_{n\to\infty} \langle (X_{T^n_A} - X_{S^n_A})y, z\rangle \\
&= \lim_{n\to\infty} \langle 1_A(X_{T^n} - S_{X^n})y, z\rangle \\
&= \lim_{n\to\infty} \langle h(X_{T^n} - X_{S^n}), z\rangle
\end{aligned}$$

Then the equality holds for any $\mathcal{F}_S$-step function h. □

Proposition 43. *Let $S \le T$ be stopping times and assume that either*

(i) $h : \Omega \to \mathbb{R}$ *is a simple, $\mathcal{F}_S$-measurable function and* $H \in L^1_{F,G}(X)$,

or

(ii) *The measure I_X is $\sigma-$additive, $h : \Omega \to F$ is a simple, $\mathcal{F}_S$-measurable function and* $H \in L^1_{\mathbb{R},E}(X)$.

If $\int 1_{(S,T]}H\, dI_X \in L^p_G$ in case (i) and $\int 1_{(S,T]}H\, dI_X \in L^p_E$ in case (ii) then

$$\int h1_{(S,T]}H\, dI_X = h \int 1_{(S,T]}H\, dI_X.$$

Proof. Assume first hypothesis (i). Let (T^n) and (S^n) be two sequences of simple stopping times such that $T^n \downarrow T$, $S^n \downarrow S$ and $S^n \le T^n$. Assume $H = 1_{(s,t]\times A}y$ with $A \in \mathcal{F}_s$ and $y \in F$. Then $T^n \wedge t \downarrow T \wedge t$, $S^n \wedge s \downarrow S \wedge s$. Let $z \in (L^p_G)^*$. Then

$$\langle \int h1_{(S,T]}H\, dI_X, z\rangle = \langle \int h1_A y 1_{(S\vee s, T\wedge t]}\, dI_X, z\rangle,$$

where the bracket represents the duality between L^p_G and $(L^p_G)^*$. By (4.2), for the simple $\mathcal{F}_{S\vee s}$-measurable function $h1_A y$ and the stopping times $(S \vee s) \le (T \wedge t)$ we have

$$\begin{aligned}\langle h\int 1_{(S,T]}H\,dI_X, z\rangle &= \langle\int 1_{(S,T]}H dI_X, hz\rangle\\ &= \langle\int 1_{(S\vee s,T\wedge t]}1_A y dI_X, hz\rangle\\ &= \lim\langle 1_A y(X_{T^n\wedge t} - X_{S^n\vee s}), hz\rangle\\ &= \lim\langle h1_A y(X_{T^n\wedge t} - X_{S^n\vee s}), z\rangle\\ &= \langle\int h1_A y 1_{(S\vee s,T\wedge t]} dI_X, z\rangle\\ &= \langle\int h1_A y 1_{(s,t]}1_{(S,T]} dI_X, z\rangle\\ &= \langle\int hH1_{(S,T]} dI_X, z\rangle\end{aligned}$$

If $H = 1_{\{0\}\times A}y$ with $A \in \mathcal{F}_0$ and $y \in F$, since $1_{(S,T]}1_{\{0\}\times A} = 0$ we have

$$\langle h\int 1_{(S,T]}H\,dI_X, z\rangle = 0 = \langle\int hH1_{(S,T]}dI_X, z\rangle.$$

It follows that for any $B \in \mathcal{R}$ we have

$$\langle\int h1_{(S,T]}1_B y\,dI_X, z\rangle = \langle h\int 1_{(S,T]}1_B y\,dI_X, z\rangle. \qquad (*)$$

The class $\mathcal{M}$ of sets $B \in \mathcal{P}$ for which the above equality holds for all $z \in (L^p_G)^*$ is a monotone class: in fact, let B_n be a monotone sequence of sets from $\mathcal{M}$ and let $B = \lim B_n$. For each n we have

$$\int h1_{(S,T]}1_{B_n}yd(I_X)_z = \langle h\int 1_{(S,T]}1_{B_n}ydI_X, z\rangle.$$

Since $h1_{(S,T]}1_{B_n}y$ is a sequence of bounded functions converging to $h1_{(S,T]}1_B y$ (h is a step-function) with $|h1_{(S,T]}1_{B_n}y| \leq |h||y|$, we can apply Lebesgue Theorem and conclude that $\int h1_{(S,T]}1_{B_n}yd(I_X)_z \to \int h1_{(S,T]}1_B yd(I_X)_z$. Using the same reasoning we can conclude that $\int 1_{(S,T]}1_{B_n}yd(I_X)_{hz} \to \int 1_{(S,T]}1_B yd(I_X)_{hz}$. hence we have

$$\langle\int h1_{(S,T]}1_B y\,dI_X, z\rangle = \lim_n\langle\int h1_{(S,T]}1_{B_n}y\,dI_X, z\rangle$$

$$= \lim_n \langle h \int 1_{(S,T]} 1_{B_n} y \, dI_X, z \rangle$$

$$= \langle h \lim_n \int 1_{(S,T]} 1_{B_n} y \, dI_X, z \rangle$$

$$= \langle h \int 1_{(S,T]} 1_B y \, dI_X, z \rangle$$

Since the class $\mathcal{M}$ of sets satisfying equality (*) is a monotone class containing $\mathcal{R}$ we conclude that the equality (*) is satisfied by all $B \in \mathcal{P}$.

It follows that for any predictable, simple process H and for each $z \in (L^p_G)^*$ we have

$$\langle \int h 1_{(S,T]} H \, dI_X, z \rangle = \langle h \int 1_{(S,T]} H \, dI_X, z \rangle \tag{**}$$

Consider now the general case. If $H \in L^1_{F,G}(X)$, then there is a sequence (H^n) of simple, predictable processes such that $H^n \to H$ and $|H^n| \leq |H|$. We apply Lebesgue's Theorem and deduce that

$$\int h 1_{(S,T]} H^n \, d(I_X)_z \to \int h 1_{(S,T]} H \, d(I_X)_z, \tag{1}$$

and

$$\int 1_{(S,T]} H^n \, d(I_X)_{hz} \to \int 1_{(S,T]} H \, d(I_X)_{hz}. \tag{2}$$

By equality (**) for each n we have

$$\begin{aligned} &\int h 1_{(S,T]} H^n \, d(I_X)_z \\ &\qquad = \langle \int h 1_{(S,T]} H^n \, dI_X, z \rangle = \langle h \int 1_{(S,T]} H^n \, dI_X, z \rangle \\ &\qquad = \langle \int 1_{(S,T]} H^n \, dI_X, hz \rangle \\ &\qquad = \int 1_{(S,T]} H^n \, d(I_X)_{hz} \end{aligned}$$

By (1) and (2) we deduce that

$$\int h1_{(S,T]}H\,d(I_X)_z = \int 1_{(S,T]}H\,d(I_X)_{hz},$$

which is equivalent to

$$\langle \int h1_{(S,T]}H\,dI_X, z\rangle = \langle \int 1_{(S,T]}H\,dI_X, hz\rangle.$$

We conclude that

$$\int h1_{(S,T]}H\,dI_X = h\int 1_{(X,T]}H\,dI_X, \text{ a.e.}$$

Assume now hypothesis (ii). Since the measure I_X is $\sigma-$additive the process X is summable. Then observe that the assumptions of (ii) are the same as the assumptions in Proposition 11.5 (ii) of [Din00]. Hence

$$\int h1_{(S,T]}H\,dI_X = h\int 1_{(X,T]}H\,dI_X,$$

which concludes our proof. □

Proposition 44. *Let $X : \mathbb{R}\times\Omega \to E \subset L(F,G)$ be a p-additive summable process relative to (F,G) and T a stopping time.*

a) For every $z \in (L^p_G)^*$ and every $B \in \mathcal{P}$ we have:

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]).$$

b) The measure $I_{X^T} : \mathcal{R} \to L^p_E$ has finite semivariation relative to (F, L^p_G)

c) If T is a simple stopping time then the process X^T is summable.

Proof. a) First we prove that if T and S are simple stopping times then we have $I_X((S,T]) = X_T - X_S$.

Assume that T is a simple stopping time of the form

$$T = \sum_{1\le i\le n} 1_{A_i} t_i,$$

with $0 < t_i \le \infty$, $t_i \neq t_j$ for $i \neq j$, $A_i \in \mathcal{F}_{t_i}$ are mutually disjoint and $\bigcup_{1\le i\le n} A_i = \Omega$. Then $[0,T] = \bigcup_{1\le i\le n}[0,t_i] \times A_i$ is a disjoint union. Hence $I_X([0,T]) = \sum_i I_X([0,t_i] \times A_i) = \sum_i 1_{A_i} X_{t_i} = X_T$. Since $(S,T] = [0,T] - [0,S]$ and I_X is an additive measure, we have $I_X((S,T]) = I_X([0,T]) - I_X([0,S]) = X_T - X_S$.

Next observe that if T is a simple stopping time then $T\wedge t$ is also a simple stopping time. In fact, if $T = \sum_{1\le i\le n} 1_{A_i} t_i$ then $T \wedge t = \sum_{i:t_i<t} 1_{A_i} t_i + \sum_{i:t_i\ge t} 1_{A_i} t$ which is a simple stopping time.

Now we prove that for $B \in \mathcal{R}$ we have

$$I_{X^T}(B) = I_X([0,T] \cap B).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$I_{X^T}(\{0\} \times A) = 1_A X_0 = I_X(\{0\} \times A) = I_X([0,T] \cap (\{0\} \times A)).$$

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$\begin{aligned} I_{X^T}((s,t] \times A) &= 1_A(X_t^T - X_s^T) = 1_A(X_{T\wedge t} - X_{T\wedge s}) \\ &= 1_A(I_X((T\wedge s, T\wedge t]) = 1_A \int 1_{(s,t]} 1_{[0,T]} dI_X \\ &= \int 1_A 1_{(s,t]} 1_{[0,T]} dI_X = I_X([0,T] \cap ((s,t] \times A)). \qquad (*) \end{aligned}$$

We used the above Proposition 43 with $h = 1_A$, $(S,T] = (s,t]$ and $H = 1_{[0,T]}$.

Next we consider the general case, with T a stopping time.

For $A \in \mathcal{F}_0$ we have

$$I_{X^T}(\{0\} \times A) = 1_A X_0 = I_X(\{0\} \times A) = I_X([0,T] \cap (\{0\} \times A)).$$

Let $y \in F$ and $z \in (L_G^p)^*$. We have

$$\begin{aligned} &\langle (I_{X^T})_z(\{0\} \times A), y\rangle = \langle I_{X^T}(\{0\} \times A)y, z\rangle \\ =&\langle I_X([0,T] \cap (\{0\} \times A))y, z\rangle = \langle (I_X)_z([0,T] \cap (\{0\} \times A)), y\rangle \qquad (1) \end{aligned}$$

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$I_{X^T}((s,t] \times A) = 1_A(X_t^T - X_s^T) = 1_A(X_{T\wedge t} - X_{T\wedge s}) \tag{2}$$

Let T_n be a sequence of simple stopping times such that $T_n \downarrow T$. Let $y \in F$ and $z \in (L_G^p)^*$. We have by (2):

$$\begin{aligned}\langle (I_{X^T})_z((s,t] \times A), y\rangle &= \langle I_{X^T}((s,t] \times A)y, z\rangle = \langle 1_A(X_{T\wedge t} - X_{T\wedge s})y, z\rangle \\ &= \lim_{n\to\infty} \langle 1_A(X_{T_n\wedge t} - X_{T_n\wedge s})y, z\rangle.\end{aligned}$$

By (*) we have:

$$\begin{aligned}&\lim_{n\to\infty} \langle 1_A(X_{T_n\wedge t} - X_{T_n\wedge s})y, z\rangle = \lim_{n\to\infty} \langle I_X([0,T_n] \cap ((s,t] \times A))y, z\rangle \\ &= \lim_{n\to\infty} \langle (I_X)_z([0,T_n] \cap ((s,t] \times A)), y\rangle = \langle (I_X)_z([0,T] \cap ((s,t] \times A)), y\rangle \qquad (3)\end{aligned}$$

since $(I_X)_z$ is σ-additive. By (1) and (3) and the fact that $(I_{X^T})_z$ is σ-additive we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{R} \tag{4}$$

Since $(I_X)_z$ is σ-additive we deduce that $(I_{X^T})_z$ is σ-additive, hence it can be extended to a σ-additive measure on $\mathcal{P}$. Since $(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T])$ for all $B \in \mathcal{R}$ we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{P},$$

b) Let A be a set in $\mathcal{R}$. By Proposition 4.15 in [Din00] we have $svar_{F,L_G^p} I_{X^T}(A) < \infty$ if and only if $var(I_{X^T})_z(A) < \infty$ for each $z \in (L_G^p)^*$. But

$$\begin{aligned}\sup_{z\in((L_G^p)^*)_1} var(I_{X^T})_z(A) &= \sup_{z\in((L_G^p)^*)_1} var(I_X)_z(A \cap [0,T]) \\ &= svar_{F,L_G^p} I_X(A \cap [0,T]) < \infty,\end{aligned}$$

and Assertion b) is proved.

c) Assume T is a simple stopping time. By the equality (*) we have

$$I_{X^T}(B) = I_X([0,T] \cap B), \text{ for } B \in \mathcal{R}.$$

Since X is p-additive summable relative to (F,G), I_X has a canonical additive extension $I_X : \mathcal{P} \to L^p_G$. The equality

$$I_{X^T}(A) = I_X([0,T] \cap A), \text{ for } A \in \mathcal{P},$$

defines an additive extension of I_{X^T} to $\mathcal{P}$. Since the measure I_X has finite semivariation relative to (F, L^p_G) (X is additive summable), the measure I_{X^T} has finite semivariation relative to (F, L^p_G) also. Moreover, for each $z \in (L^p_G)^*$, by Assertion a), the measure $(I_{X^T})_z$ defined on $\mathcal{P}$ is $\sigma-$additive. Therefore X^T is additive summable. We have $|(I_{X^T})_z|(A) = |(I_X)_z|([0,T] \cap A)$ for $A \in \mathcal{P}$ since $|(I_X)_z|$ is the canonical extension of its restriction on $\mathcal{R}$. Then $|(I_{X^T})_z|$ is the canonical extension of its restriction to $\mathcal{R}$. it follows that I_{X^T} is the canonical extension of its restriction to $\mathcal{R}$. □

The next theorem gives the relationship between the stopped stochastic integral and the integral of the process $1_{[0,T]}H$. The same type of relation was proved in Theorem 11.6 in [Din00].

Theorem 45. *Let $H \in L^1_{F,G}(X)$ and let T be a stopping time. Then $1_{[0,T]}H \in L^1_{F,G}(X)$ and*

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

Proof. Suppose first that T is a simple stopping time of the form

$$T = \sum_{1 \le i \le n} 1_{A_i} t_i$$

with $0 \leq t_1 \leq t_2 \leq \dots t_n \leq +\infty$, $A_i \in \mathcal{F}_{t_i}$ mutually disjoint and with union Ω. Then for $t \geq 0$ we have

$$(H \cdot X)^T_t(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega).$$

In fact, for $\omega \in \Omega$ there is $1 \leq i \leq n$ such that $\omega \in A_i$. Then $T(\omega) = t_i$, hence

$$(H \cdot X)^T_t(\omega) = (H \cdot X)_{t_i \wedge t}(\omega).$$

On the other hand

$$(1_{[0,T]}H) \cdot X)_t(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega).$$

In fact,

$$\begin{aligned}
(\int_{[0,t]} 1_{[0,T]} H dI_X)(\omega) &= (\int_{[0,t]} \sum_{1 \leq i \leq n} 1_{[0,t_i]} 1_{A_i} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} (\int_{[0,t_i \wedge t]} 1_{A_i} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} (\int_{[0,\infty]} H 1_{A_i} dI_X)(\omega) - \sum_{1 \leq i \leq n} (\int_{(t_i \wedge t, \infty]} 1_{A_i} H dI_X)(\omega) \\
&= (\int_{[0,\infty]} H dI_X)(\omega) - \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{(t_i,\infty]} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{[0,\infty]} H dI_X)(\omega) - \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{(t_i,\infty]} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{[0,t_i \wedge t]} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega),
\end{aligned}$$

where the 4th equality is obtained by applying Proposition 43, with $h = 1_{A_i}$.

Hence, for T simple, we have$1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

Now choose T arbitrary. Then there is a decreasing sequence (T_n) of simple stopping times, such that $T_n \downarrow T$.

Note first that since $(H \cdot X)$ is cadlag we have

$$(H \cdot X)^{T_n} \to (H \cdot X)^T. \tag{1}$$

Moreover for $t \geq 0$ we have $1_{[0,T_n\wedge t]}H \downarrow 1_{[0,T_n\wedge t]}H$ pointwise. Since $1_{[0,T_n\wedge t]}H \in L^1_{F,G}(X)$, for each $(z \in L^p_G)^*$ we have $1_{[0,T_n\wedge t]}H \in L^1_F(|(I_X)_z|)$, hence

$$\langle \int 1_{[0,T_n\wedge t]}HdI_X, z\rangle = \int 1_{[0,T_n\wedge t]}Hd(I_X)_z \to \int 1_{[0,T\wedge t]}Hd(I_X)_z = \langle \int 1_{[0,T\wedge t]}HdI_X, z\rangle.$$

By Theorem 36 we conclude that $\int 1_{[0,T\wedge t]}HdI_X = \int_{[0,t]} 1_{[0,T]}HdI_X \in L^p_G$ and

$$\int 1_{[0,T_n\wedge t]}HdI_X \to \int 1_{[0,T\wedge t]}HdI_X,$$

or

$$\int_{[0,t]} 1_{[0,T_n]}HdI_X \to \int_{[0,t]} 1_{[0,T]}HdI_X.$$

Since for each n we have $(1_{[0,T_n]}H \cdot X)_t = (H \cdot X)^{T_n}_t$, by (1) we deduce that $\int_{[0,t]} 1_{[0,T]}HdI_X = (H \cdot X)^T_t$. Hence the mapping $t \mapsto \int_{[0,t]} 1_{[0,T]}HdI_X$ is cadlag, from which we conclude that $1_{[0,T]}H \in L^1_{F,G}(X)$. Moreover

$$(1_{[0,T]}H \cdot X)_t = (H \cdot X)_{T\wedge t} = (H \cdot X)^T_t.$$

$\square$

The next corollary is a useful particular case of the previous theorem:

Corollary 46. *For every stopping time T we have*

$$1_{[0,T]} \cdot X = X^T.$$

Proof. Taking $H = 1 \in L^1_{F,G}(X)$ and applying Theorem 45 we conclude that $1_{[0,T]} \cdot X = X^T$. $\square$

The following theorem gives the same type of results as Theorem 11.8 in[Din00].

Theorem 47. *Let* $S \leq T$ *be stopping times and assume that either*

(i) $h : \Omega \to \mathbb{R}$ *is a simple,* $\mathcal{F}_S$*-measurable function and* $H \in L^1_{F,G}(X)$*,*

or

(ii) *The measure* I_X *is* $\sigma-$*additive,* $h : \Omega \to F$ *is a simple,* $\mathcal{F}_S$*-measurable function and* $H \in L^1_{\mathbb{R},E}(X)$*.*

Then $1_{(S,T]}H$ *and* $h1_{(S,T]}H$ *are integrable with respect to* X *and*

$$(h1_{(S,T]}H) \cdot X = h[(1_{(S,T]}H) \cdot X].$$

Proof. Note that

$$1_{(S,T]}H = 1_{[0,T]}H - 1_{[0,S]}H$$

Assume first the case (i). Applying Theorem 45 for $1_{[0,T]}H$ and $1_{[0,S]}H$ we conclude that $1_{(S,T]}H \in L^1_{F,G}(X)$.

If for each $t \geq 0$ we apply Proposition 43, we obtain

$$\int_{[0,t]} h1_{(S,T]}HdI_X = h\int_{[0,t]} 1_{(S,T]}HdI_X.$$

Since $1_{(S,T]}H \in L^1_{F,G}(X)$ we deduce that $h1_{(S,T]}H \in L^1_{F,G}(X)$ and

$$((h1_{(S,T]}H) \cdot X)_t = h((1_{(S,T]}H) \cdot X)_t$$

which concludeds the proof of case (i). Case (ii) is treated similarly. □

4.3 The Integral $\int HdI_{X^T}$

In this section we define the set of processes integrable with respect to the measure I_{X^T} with finite semivariation relative to the pair (F, L^p_G).

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be a cadlag, adapted process and assume X is p-additive summable relative to (F,G).

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F, L^p_G)$ with bounded semivariation $\tilde{I}_{F,G}$ relative to (F, L^p_G), such that each of the measures $(I_X)_z$ with $z \in (L^p_G)^*$ is σ-additive.

To simplify the notations denote $m = I_{X^T}$. We proved in the previous proposition that the measure $m : \mathcal{R} \to L^p_E \subset L(F, L^p_G)$ has bounded semivariation relative to (F, L^p_G), on $\mathcal{R}$, and for each $z \in (L^p_G)^*$ the measures m_z, is σ-additive. In order for the process X^T to be additive summable we need the measure $m : \mathcal{R} \to L^p_E$ to have an extension $m : \mathcal{P} \to L^p_E$ with finite semivariation and such that each of the measures m_z with $z \in (L^p_G)^*$ is $\sigma-$additive. Applying Theorem 7 from Bongiorno–Dinculeanu, citeBD2001, the measure m has a unique canonical extension $m : \mathcal{P} \to (L^p_E)^{**}$, with bounded semivariation such that for each $z \in (L^p_G)^*$ the measure m_z, is σ-additive and has bounded variation $|m_z|$, therefore X^T is summable.

Then we have

$$\tilde{m}_{F,L^p_G} = \sup\{|m_z| : z \in (L^p_G)^*, \|z\|_q \leq 1\}.$$

We denote by $\mathcal{F}_{F,G}(X^T)$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to F$ such that

$$\tilde{m}_{F,G}(H) = \tilde{m}_{F,L^p_G}(H) = \sup\{\int |H| d|m_z| : \|z\| \leq 1\} < \infty.$$

Let $H \in \mathcal{F}_{F,G}(X^T)$; then $H \in L^1_F(|m_z|)$ for every $z \in (L^p_G)^*$, hence the integral $\int H dm_z$ is defined and is a scalar. The mapping $z \mapsto \int H dm_z$ is a linear continuous functional on $(L^p_G)^*$, denoted $\int H dm$. Therefore, $\int H dm \in (L^p_G)^{**}$,

$$\langle \int Hdm, z\rangle = \int Hdm_z, \text{ for } z \in (L^p_G)^*.$$

We denote by $L^1_{F,G}(X^T)$ the set of processes $H \in \mathcal{F}_{F,G}(I^T_X)$ satisfying the following two conditions:

a) $\int_{[0,t]} Hdm \in L^p_G$ for every $t \in \mathbb{R}_+$;

b) The process $(\int_{[0,t]} Hdm)_{t \geq 0}$ has a cadlag modification.

Theorem 48. *Let $X : \mathbb{R} \to E \subset L(F,G)$ be a p-additive summable process relative to (F,G) and T a stopping time.*

a) We have $H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have:

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X.$$

b) We have $H \in L^1_{F,G}(X^T)$ iff $1_{[0,T]}H \in L^1_{F,G}(X)$ and in this case we have:

$$H \cdot X^T = (1_{[0,T]}H) \cdot X.$$

If $H \in L^1_{F,G}(X)$, then $H \in L^1_{F,G}(X^T), 1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(H \cdot X)^T = H \cdot X^T = (1_{[0,T]}H) \cdot X.$$

Proof. a) Define $m : \mathcal{R} \to E$ by $m(B) = I_{X^T}(B)$ for $B \in \mathcal{R}$. We proved in Theorem 44 (a) that for every $z \in (L^p_G)^*$ we have

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{R}. \qquad (*)$$

Since $(I_X)_z((\cdot) \cap [0,T])$ is a $\sigma-$additive measure, with bounded variation on $\mathcal{P}$ satisfying (*) and since $\mathcal{P}$ is the $\sigma-$algebra generated by $\mathcal{R}$, by the uniqueness theorem 7.4 in [Din00] we conclude that

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{P}.$$

Let $H \in \mathcal{F}_{F,G}(X^T) = \bigcap_{\|z\|_q \le 1, z \in (L^p_G)^*} L^1_F(m_z)$. From the previous equality we deduce that

$$\int H dm_z = \int 1_{[0,T]} H d(I_X)_z,$$

therefore

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X,$$

and this is the equality in Assertion a).

b) To prove Assertion b) we replace H with $1_{[0,t]}H$ in the previous assertion and deduce that $1_{[0,t]}H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,t]}1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have

$$\int_{[0,t]} H dI_{X^T} = \int_{[0,t]} 1_{[0,T]} H dI_X.$$

It follows that $H \in L^1_{F,G}(X^T)$ iff $1_{[0,T]}H \in L^1_{F,G}(X)$ and in this case we have

$$(H \cdot X^T)_t = ((1_{[0,T]}H) \cdot X)_t.$$

If now $H \in L^1_{F,G}(X)$, then, from Theorem 45 we deduce that

$1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

□

4.4 Convergence Theorems

Assume X is p-additive summable relative to (F, G). In this section we shall prove several convergence theorems.

Lemma 49. *Let (H^n) be a sequence in $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(X)$. Then there is a subsequence (r_n) such that*

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t = \int_{[0,t]} H dI_X, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval.

Proof. Since H^n is a convergent sequence in $\mathcal{F}_{F,G}(X)$ there is a subsequence H^{r_n} of (H^n) such that

$$\tilde{I}_{F,G}(H^{r_n} - H^{r_{n+1}}) \leq 4^{-n}, \text{ for each } n.$$

Let $t_0 > 0$. Define the stopping time

$$u_n = \inf\{t : |(H^{r_n} \cdot X)_t - (H^{r_{n+1}} \cdot X)_t| > 2^{-n}\} \wedge t_0.$$

By Theorem 45 applied to the stopping time u_n, we obtain

$$(H^{r_n}\cdot X)_{u_n} = (H^{r_n}\cdot X)^{u_n}_\infty = ((1_{[0,u_n]}H^{r_n})\cdot X)_\infty = \int_{[0,u_n]} H^{r_n} dI_X,$$

hence

$$\begin{aligned}
&E(|(H^{r_n}\cdot X)_{u_n} - (H^{r_{n+1}}\cdot X)_{u_n}|) = E(|\int_{[0,u_n]} H^{r_n} dI_X - \int_{[0,u_n]} H^{r_{n+1}} dI_X|)\\
&= E(|\int_{[0,u_n]}((H^{r_n} - H^{r_{n+1}}) dI_X|) = (\|\int_{[0,u_n]}(H^{r_n} - H^{r_{n+1}}) dI_X\|_{L^1_G}\\
&\leq \|\int_{[0,u_n]}(H^{r_n} - H^{r_{n+1}}) dI_X\|_{L^p_G} \leq \tilde{I}_{F,G}(H^{r_n} - H^{r_{n+1}}) \leq 4^{-n}. \qquad (*)
\end{aligned}$$

Next let $G_n = \{u_n < t_0\}$ and $G_0 = \limsup G_n$. We shall prove that $P(G_0) = 0$.

Let $\omega \in G_n$. Then we have $u_n(\omega) < t_0$ hence $u_n = \inf\{t : |(H^{r_n}\cdot X)_t - (H^{r_{n+1}}\cdot X)_t| > 2^{-n}\} < t_0$. Therefore there is a sequence $t_i \downarrow u_n(\omega)$ with $t_i < t_0$ such that $|(H^{r_n}\cdot X)_{t_i}(\omega) - (H^{r_{n+1}}\cdot X)_{t_i}(\omega)| > 2^{-n}$ for each i; by right continuity of the stochastic integral we deduce that

$$|(H^{r_n}\cdot X)_{u_n}(\omega) - (H^{r_{n+1}}\cdot X)_{u_n}(\omega)| \geq 2^{-n};$$

hence

$$E(|(H^{r_n}\cdot X)_{u_n} - (H^{r_{n+1}}\cdot X)_{u_n}|) \geq \int_{G_n} |(H^{r_n}\cdot X)_{u_n} - (H^{r_{n+1}}\cdot X)_{u_n}| dP \geq 2^{-n}P(G_n)$$

Therefore $P(G_n) \leq 2^n E(|Z^n_{u_n} - Z^{n+1}_{u_n}|) \leq 2^{-n}$, by (*). Hence $P(G_0) = 0$.

For $\omega \notin G_0$, there is an N such that, if $n \geq N$, we have $\omega \notin G_n$, hence $u_n(\omega) = t_0$. Thus

$$\sup_{t<t_0} |(H^{r_n}\cdot X)_{u_n}(\omega) - (H^{r_{n+1}}\cdot X)_{u_n}(\omega)| \leq 2^{-n}.$$

It follows that for $\omega \notin G_0$, the sequence $((H^{r_n}\cdot X)_t(\omega))$ is Cauchy in G, uniformly for $t < t_0$. Using the inequality (*) we conclude that $(H^{r_n}\cdot X)_t$ is a Cauchy sequence in L^p_G uniformly for $t < t_0$. Define the process $Y : [0, t_0) \to G$ by

$$Y_t(\omega) = \lim_n (H^{r_n} \cdot X)_t(\omega), \text{ if } \omega \notin G_0,$$

and $Y_t(\omega) = 0$ if $\omega \in G_0$.

Since $H^n \to H$ in $\mathcal{F}_{F,G}(X)$, for every $t \geq 0$ we have also $1_{[0,t]}H^n \to 1_{[0,t]}H$ in $\mathcal{F}_{F,G}(X)$. Since the stochastic integral is continuous, we deduce that

$$(H^n \cdot X)_t = \int_{[0,t]} H^n dI_X \to \int_{[0,t]} H dI_X, \text{ in } (L^p_{G^*})^*.$$

Since $H^n \in L^1_{F,G}(X)$ we have $\int_{[0,t]} H^n dI_X \in L^p_G$ and

$$(H^n \cdot X)_t \to \int_{[0,t]} H dI_X, \text{ in } L^p_G.$$

It follows that the two limits of $((H^{r_n} \cdot X)_t)$ are equal a.s. Hence

$$\int_{[0,t]} H dI_X = Y_t \text{ a.s. for } t < t_0.$$

It follows that

$$(H^{r_n} \cdot X)_t \to \int_{[0,t]} H dI_X,$$

uniformly on every bounded time interval. □

Theorem 50. *Let (H^n) be a sequence from $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(X)$. Then:*

a) $H \in L^1_{F,G}(X)$.

b) $(H^n \cdot X)_t \to (H \cdot X)_t$, *in* L^p_G, *for* $t \in [0, \infty]$.

c) *There is a subsequence* (r_n) *such that*

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval.

Proof. For every $t \geq 0$ we have $1_{[0,t]}H^n \to 1_{[0,t]}H$ in $\mathcal{F}_{F,G}(X)$. Since the integral is continuous, we deduce that

$$(H^n \cdot X)_t = \int_{[0,t]} H^n dI_X \to \int_{[0,t]} H dI_X, \text{ in } (L^p_{G^*})^*.$$

Since $H^n \in L^1_{F,G}(X)$ we have $\int_{[0,t]} H^n dI_X \in L^p_G$ and

$$(H^n \cdot X)_t \to \int_{[0,t]} H dI_X, \text{ in } L^p_G.$$

From the previous lemma we deduce that there is a subsequence (H^{r_n}) such that

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval. Since $(H^{r_n} \cdot X)$ are cadlag it follows that the limit is also cadlag, hence $H \in L^1_{F,G}(X)$ which is Assertion a). Hence

$$(H \cdot X)_t = \int_{[0,t]} H dI_X, \text{ a.s.}$$

and therefore $(H^n \cdot X)_t \to (H \cdot X)_t$, in L^p_G, which is Assertion b). Also observe that for the above susequence (H^{r_n}) we have

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval. □

We can restate Theorem 50 as:

Corollary 51. *$L^1_{F,G}(X)$ is complete.*

Next we state an uniform convergence theorem. Uniform convergence implies convergence in $L^1_{F,G}(X)$.

Theorem 52. *Let (H^n) be a sequence from $\mathcal{F}_{F,G}(X)$. If $H^n \to H$ pointwise uniformly then $H \in \mathcal{F}_{F,G}(X)$ and $H^n \to H$ in $\mathcal{F}_{F,G}(X)$.*

If, in addition, for each n, H^n is integrable, i.e. $H^n \in L^1_{F,G}(X)$ then

a) *$H \in L^1_{F,G}(X)$ and $H^n \to H$ in $L^1_{F,G}(X)$;*

b) *For every* $t \in [0, \infty]$ *we have* $(H^n \cdot X)_t \to (H \cdot X)_t$, *in* L^p_G.

c) *There is a subsequence* (r_n) *such that* $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$, *a.s. as* $n \to \infty$, *uniformly on any bounded interval.*

Proof. Assertion a) is immediate. Assertions b), c) and d) follow from Theorem 50. □

Now we shall state Vitali and Lebesgue-type Convergence Theorems. They are direct consequences of the convergence Theorem 50 and of the uniform convergence Theorem 52.

Theorem 53. (Vitali). *Let* (H^n) *be a sequence from* $\mathcal{F}_{F,G}(X)$ *and let* H *be an* F*-valued, predictable process. Assume that*

(i) $\tilde{I}_{F,G}(H^n 1_A) \to 0$ *as* $\tilde{I}_{F,G}(A) \to 0$, *uniformly in* n

and that any one of the conditions (ii) *or* (iii) *below is true:*

(ii) $H^n \to H$ *in* $\tilde{I}_{F,G}$*-measure;*

(iii) $H^n \to H$ *pointwise and* $I_{F,(L^p_G)^*}$ *is uniformly* σ*-additive (this is the case if* H^n *are real-valued, i.e.,* $F = \mathbb{R}$*).*

Then:

a) $H \in \mathcal{F}_{F,G}(X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$.

Conversely, if $H^n, H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$, *then conditions* (i) *and* (ii) *are satisfied.*

Under the hypotheses (i) *and* (ii) *or* (iii), *assume, in addition, that* $H^n \in L^1_{F,G}(X)$ *for each* n. *Then*

b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$ *in* $L^1_{F,G}(X)$*;*

c) *For every* $t \in [0, \infty]$ *we have* $(H^n \cdot X)_t \to (H \cdot X)_t$, *in* L^p_G*;*

d) *There is a subsequence* (r_n) *such that* $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$, *a.s., as* $n \to \infty$, *uniformly on any bounded interval.*

Theorem 54. (Lebesgue). *Let* (H^n) *be a sequence from* $\mathcal{F}_{F,G}(X)$ *and let* H *be an* F*-valued predictable process. Assume that*

(i) *There is a process* $\phi \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, I_{F,G})$ *such that*

$$|H^n| \leq \phi \textit{ for each } n;$$

and that any one of the conditions (ii) *or* (iii) *below is true:*

(ii) $H^n \to H$ *in* $\tilde{I}_{F,G}$*-measure;*

(iii) $H^n \to H$ *pointwise and* $I_{F,L^q_{G^*}}$ *is uniformly* σ*-additive (this is the case if* H^n *are real valued, i.e.,* $F = \mathbb{R}$*).*

Then:

a) $H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$.

Assume, in addition that $H^n \in L^1_{F,G}(X)$ *for each* n. *Then*

b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$ *in* $L^1_{F,G}(X)$;

c) *For every* $t \in [0, \infty]$ *we have* $(H^n \cdot X)_t \to (H \cdot X)_t$, *in* L^p_G;

d) *There is a subsequence* (r_n) *such that* $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$, *a.s., as* $n \to \infty$, *uniformly on any bounded interval.*

4.5 Summability of the Stochastic Integral

Assume X is p-additive summable relative to (F, G). In this section we are studying the additive summability of the stochastic integral $H \cdot X$ for $F-$valued processes H.

If H is a real valued processes then in order for the stochastic integral $H \cdot X$ to be defined we need each of the measure $(I_X)_z$, for $z \in (L^p_E)^*$, to be $\sigma-$additve, hence the measure I_X would be $\sigma-$additive. Therefore the process X would be summable. In this case the summability of the stochastic integral is proved in Theorem 13.1 of [Din00].

Theorem 55. *[13.1, [Din00]] Let* $H \in \mathcal{F}_{\mathbb{R}}(\tilde{I}_{F,G})$. *Assume that* $H \in L^1_{\mathbb{R},E}(X)$ *and* $\int_{[0,t]} H dI_X \in L^p_E$ *for* $t \geq 0$ *Then:*

a) *$H \cdot X$ is p-summable relative to (F, G) and*

$$dI_{H \cdot X} = d(HI_X)$$

where HI_X is the measure defined by

$$(HI_X)(A) = \int_A H dI_X, \text{ for } A \in \mathcal{P}.$$

b) *For any predictable process $K \geq 0$ we have*

$$(\tilde{I}_{H \cdot X})_{F,G}(K) = (\tilde{I}_X)_{F,G}(KH).$$

c) *$K \in L^1_{F,G}(H \cdot X)$ iff $KH \in L^1_{F,G}(X)$, and in this case we have*

$$K \cdot (H \cdot X) = (KH) \cdot X.$$

The next theorem shows that if H is $F-$valued then the measure $I_{H \cdot X}$ is $\sigma-$additive even if I_X is just additive. For the proof of the theorem we need the following proposition:

Proposition 56 (5.48(c)[Din00]). *If $g \in \mathcal{F}_{F,G}(m)$ and $\int_A g\, dm \in G$ for every $A \in \Sigma$, then the measure gm is $\sigma-$additive.*

Theorem 57. *Let $H \in L^1_{F,G}(X)$ be such that $\int_A H dI_X \in L^p_G$ for $A \in \mathcal{P}$. Then the measure $I_{H \cdot X} : \mathcal{R} \to L^p_G$ has a $\sigma-$additive extension $I_{H \cdot X} : \mathcal{P} \to L^p_G$ to $\mathcal{P}$.*

Proof. We first note that $H \cdot X : \mathbb{R}_+ \times \Omega \to G = L(\mathbb{R}, G)$ is a cadlag adapted process with $(H \cdot X)_t \in L^p_G$ for $t \geq 0$ (by the definition of $H \cdot X$).

Since $\int_A H dI_X \in L^p_G$ for every $A \in \mathcal{P}$, by Proposition 56, with $m = I_X$ and $g = H$, we deduce that HI_X is $\sigma-$additive on $\mathcal{P}$.

Next we prove that for any predictable rectangle $A \in \mathcal{R}$ we have

$$I_{H \cdot X}(A) = \int_A H dI_X. \tag{1}$$

In fact, consider first $A = \{0\} \times B$ with $B \in \mathcal{F}_0$. Using Proposition 43 for $h = 1_B$ we have

$$\begin{aligned} I_{H\cdot X}(\{0\} \times B) &= 1_B((H \cdot X)_0) = 1_B \int_{\{0\}} H dI_X \\ &= \int_{\{0\}} 1_B H dI_X = \int_{\{0\}\times B} H dI_X; \end{aligned}$$

Let now $A = (s,t] \times B$ with $B \in \mathcal{F}_s$. Using Proposition 43 for $h = 1_B$ and $(S,T] = (s,t]$ we have

$$\begin{aligned} &I_{H\cdot X}((s,t] \times B) = 1_B((H \cdot X)_t - (H \cdot X)_s) \\ =&1_B(\int_{[0,t]} H dI_X - \int_{[0,s]} H dI_X) = 1_B \int_{(s,t]} H dI_X \\ =& \int_{(s,t]} 1_B H dI_X = \int_{(s,t]\times B} H dI_X; \end{aligned}$$

and the desired equality is proved.

Since the measure $A \mapsto \int_A H dI_X$ is $\sigma-$additive for $A \in \mathcal{P}$ it will follow that $I_{H\cdot X}$ can be extended to a σ-additive measure on $\mathcal{P}$ by the same equality

$$I_{H\cdot X}(A) = \int_A H dI_X, \text{ for } A \in \mathcal{P}. \tag{2}$$

□

The next theorem states the summability of the stochastic integral.

Theorem 58. *Let $H \in L^1_{F,G}(X)$ be such that $\int_A H dI_X \in L^p_G$ for $A \in \mathcal{P}$. Then:*

a) *$H \cdot X$ is p-summable, hence p-additive summable relative to $(\mathbb{R}, G)$ and*

$$dI_{H\cdot X} = d(HI_X).$$

b) *For any predictable process $K \geq 0$ we have*

$$(\tilde{I}_{H\cdot X})_{\mathbb{R},G}(K) \leq (\tilde{I}_X)_{F,G}(KH).$$

c) *If* K *is a real-valued predictable process and if* $KH \in L^1_{F,G}(X)$, *then* $K \in L^1_{\mathbb{R},G}(H \cdot X)$ *and we have*

$$K \cdot (H \cdot X) = (KH) \cdot X.$$

Proof. By Theorem 57 we know that the measure $I_{H \cdot X}$ is $\sigma-$additive. Therefore To prove (a) we only need to show that the extension of $I_{H \cdot X}$ to $\mathcal{P}$ has finite semivariation relative to $(\mathbb{R}, L^p_G)$.

Let $z \in (L^p_G)^*$. From the equality (2) in Theorem 57 we deduce that for every $A \in \mathcal{P}$, and we have

$$\begin{aligned}&(I_{H \cdot X})_z(A)) = \langle I_{H \cdot X}(A), z\rangle = \langle \int_A H dI_X, z\rangle \\ &= \int_A H d(I_X)_z.\end{aligned}$$

From this we deduce the inequality

$$|(I_{H \cdot X})_z|(A) \leq \int_A |H| d|(I_X)_z|, \text{ for } A \in \mathcal{P}. \tag{*}$$

Taking the supremum for $z \in (L^P_G)^*_1$ we obtain

$$\begin{aligned}\sup\{|(I_{H \cdot X})_z|(A), z \in (L^P_G)^*_1\} &\leq \sup\{\int_A |H| d|(I_X)_z|, z \in (L^P_G)^*_1\} \\ &\leq \sup\{\int |1_A H| d|(I_X)_z|, z \in (L^P_G)^*_1\}, \text{ for } A \in \mathcal{P}.\end{aligned}$$

Therefore

$$(\tilde{I}_{H \cdot X})_{\mathbb{R},G}(A) \leq (\tilde{I}_X)_{F,G}(1_A H) < \infty, \text{ for } A \in \mathcal{P}.$$

It follows that $H \cdot X$ is p-summable, hence $p-$additive summable, relative to $(\mathbb{R}, G)$ and this proves Assertion a).

Next we shall prove Assertion b). For that consider $K \geq 0$ a predictable process and (K^n) an increasing sequence of positive, simple, predictable processes such that

$K^n \uparrow K$. From inequality (*), for each n, we have

$$\int K^n d|(I_{H\cdot X})_z| \leq \int K^n |H| d|(I_X)_z|.$$

Letting $n \to \infty$ by the Monotone convergence Theorem we get, for $z \in L^q_{G^*}$

$$\int K d|(I_{H\cdot X})_z| \leq \int K |H| d|(I_X)_z|.$$

Taking the supremum for $z \in (L^p_G)^*_1$ we obtain the inequality in Assertion b).

To prove Assertion c) consider K to be a real-valued, predictable process. From b) it follows that $K \in \mathcal{F}_{\mathbb{R},G}(H \cdot X)$ iff $KH \in \mathcal{F}_{F,G}(X)$.

Assume $K \in \mathcal{F}_{\mathbb{R},G}(H \cdot X)$; therefore $KH \in \mathcal{F}_{F,G}(X)$. Moreover for each $z \in (L^p_G)^*_1$ we have $K \in L^1((I_{H\cdot X})_z)$ and $KH \in L^1_F((I_X)_z)$. Hence $1_{[0,t]}K \in L^1((I_{H\cdot X})_z)$ and $1_{[0,t]}KH \in L^1_F((I_X)_z)$ for each $t \geq 0$. From equality (*) ($(I_{H\cdot X})_z = Hd(I_X)_z$) and Theorem 2.31 a) and b) in [Din00] (with $f = 1_{[0,t]}K$, $g = H$, $m = (I_X)_z$) we have

$$\int_{[0,t]} K d(I_{H\cdot X})_z = \int_{[0,t]} KH d(I_X)_z,$$

which means

$$\langle \int_{[0,t]} K dI_{H\cdot X}, z\rangle = \langle \int_{[0,t]} KH dI_X, z\rangle,$$

which is equivalent to

$$\int_{[0,t]} K dI_{H\cdot X} = \int_{[0,t]} KH dI_X$$

From this equality we deduce that the left-hand side belongs to L^p_G and is cadlag iff the right-hand side has the same property or that $K \in L^1_{F,G}(H \cdot X)$ iff $KH \in L^1_{F,G}(X)$. Also we can rewrite it as

$$K \cdot (H \cdot X) = (KH) \cdot X,$$

which is Assertion c). □

4.6 Summability Criterion

Let $Z \subset L^q_{E^*}$ be any closed subspace norming for L^p_E. The next theorem is an improvement of the summability criterion in [Din00] where the restrictive condition $c_0 \not\subset E$ was imposed.

Theorem 59. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be an adapted, cadlag process such that $X_t \in L^p_E$ for every $t \geq 0$. Then the Assertions* a)–d) *below are equivalent.*

a) *$I_X : \mathcal{R} \to L^p_E$ has an additive extension $I_X : \mathcal{P} \to Z^*$ such that for each $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is a $\sigma-$additive on $\mathcal{P}$.*

b) *I_X is bounded on $\mathcal{R}$;*

c) *For every $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is bounded on $\mathcal{R}$;*

d) *For every $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is $\sigma-$additive and bounded on $\mathcal{R}$.*

Proof. The proof will be done as follows: b) $\Longleftrightarrow$ c) $\Longleftrightarrow$ d) and a) $\Longleftrightarrow$ d).

b) $\Longrightarrow$ c) Since I_X is bounded on $\mathcal{R}$ there is $M \geq 0$ such that for every $A \in \mathcal{R}$ we have $\|I_X(A)\|_{L^p_E} \leq M$.

Let $g \in Z$. The real valued measure $\langle I_X, g\rangle$ on $\mathcal{R}$ is defined by

$$\langle I_X, g\rangle(A) = \langle I_X(A), g\rangle, \text{ for } A \in \mathcal{R}.$$

and we have

$$|\langle I_X, g\rangle(A)| \leq \|I_X(A)\|_{L^p_E} \|g\|_q \leq M\|g\|_q \text{ for all } A \in \mathbb{R}$$

hence for each $g \in Z$, the measure $\langle I_X, g\rangle$ is bounded on $\mathcal{R}$.

c) $\Longrightarrow$ b) If we assume c), then for each $g \in Z$ we have

$$\sup\{|\langle I_X(A), g\rangle| : A \in \mathcal{R}\} < \infty.$$

For every $A \in \mathcal{R}$ consider the continuous linear functional $T_A : Z \to \mathbb{R}$ defined by

$$T_A(g) = \langle I_X(A), g\rangle \text{ for } g \in Z.$$

We can apply the Banach – Steinhauss theorem for the family $(T_A)_{A\in\mathcal{R}}$. We conclude that $\sup\{\|I_X(A)\|_{L^p_E},\ A \in \mathcal{R}\} < \infty$.

c) $\Longrightarrow$ d) Assume c), and let $g \in Z$. The real valued measure $\langle I_X, g\rangle$ is defined on $\mathcal{R}$ by

$$\langle I_X, g\rangle(A) = \langle I_X(A), g\rangle = \int \langle I_X(A), g\rangle dP, \text{ for } A \in \mathcal{R}.$$

By assumption, $\langle I_X, g\rangle$ is bounded on $\mathcal{R}$. We need to prove that the measure $\langle I_X, g\rangle$ is $\sigma-$ additive. For that consider the real-valued process $XG = (\langle X_t, G_t\rangle)_{t\geq 0}$, where $G_t = E(g|\mathcal{F}_t)$ for $t \geq 0$. Then $XG : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a cadlag, adapted process. Since $X_t \in L^p_E$ and $G_t \in L^q_{E^*}$, it follows that $(XG)_t \in L^1_{\mathbb{R}}$ for every $t \geq 0$.

Since X and G are cadlag and adapted we deduce that XG is cadlag and adapted.

We can associate to it the Doleans measure $\mu_{XG} : \mathcal{R} \to \mathbb{R}$ defined by

$$\mu_{XG}(\{0\} \times A) = E(1_A(XG)_0), \text{ for } A \in \mathcal{F}_0$$

and

$$\mu_{XG}((s,t] \times A) = E(1_A((XG)_t - (XG)_s)), \text{ for } A \in \mathcal{F}_s,$$

and then extend it by additivity to $\mathcal{R}$. We have $\mu_{XG} = \langle I_X, g\rangle$ on $\mathcal{R}$.

In fact, for $B \in \mathcal{F}_0$ we have

$$\begin{aligned}\langle I_X, g\rangle(\{0\} \times B) &= \int 1_B\langle X_0, g\rangle dP = \int 1_B(XG)_0 dP \\ &= \mu_{XG}(\{0\} \times B).\end{aligned}$$

If $s < t$ and $B \in \mathcal{F}_s$, then

$$\langle I_X, g\rangle((s,t] \times B) = \int 1_B\langle X_t - X_s, g\rangle dP$$

$$\begin{aligned} &= \int_B \langle X_t, g\rangle dP - \int_B \langle X_s, g\rangle dP \\ &= \int_B \langle X_t, G_t\rangle dP - \int_B \langle X_s, G_s\rangle dP = \int_B \langle X_t - X_s, G_s\rangle dP \\ &= \int_B (XG)_t - (XG)_s dP = \mu_{XG}((s,t] \times B). \end{aligned}$$

Since both, μ_{XG} and $\langle I_X, g\rangle$ are additive we conclude that $\mu_{XG} = \langle I_X, g\rangle$ on $\mathcal{R}$.

It follows that the real-valued measure μ_{XG} is bounded on $\mathcal{R}$. By Theorem 2.16 of [Din00], μ_{XG} has bounded variation on $\mathcal{R}$, hence XG is a quasimartingale. For each n, define the stopping time

$$T_n(\omega) = \inf\{t : |X_t| > n\}.$$

Then $T_n \uparrow \infty$ and $|X_t| \leq n$ on $[0, T_n)$. Since XG is a quasimartingale on $(0, \infty]$, we know that $(XG)_{T_n} \in L^1$ (Proposition A 3.5 in [BD87]: XG is a quasimartingale on $(0, \infty]$ iff XG is a quasimartingale on $(0, \infty)$ and $\sup_t \|XG\|_1 < \infty$.)

Moreover,

$$\begin{aligned} |(XG)_t^{T_n}| &= |(XG)_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}} \\ &\leq |X_t||G_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}} \\ &\leq n|G_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}}. \end{aligned} \tag{2}$$

Besides, since $G_t = E(g|\mathcal{F}_t)$ it follows that G is a uniformly integrable martingale.

Next we prove that the family $\{(XG)_T^{T_n}, T \text{ simple stopping time}\}$ is uniformly integrable.

In fact, note that by inequality (2) we have

$$\begin{aligned} &\int_{\{|(XG)_T^{T_n}| > p\}} |(XG)_T^{T_n}| dP \\ &\leq \int_{\{|(XG)_T^{T_n}| > p\} \cap \{T < T_n\}} n|(XG)_T^{T_n}| dP + \int_{\{|(XG)_T^{T_n}| > p\} \cap \{T \geq T_n\}} |(XG)_{T_n}| dP \end{aligned} \tag{3}$$

Now observe that

$$\{|(XG)_T| > p\} \cap \{T < T_n\} = \{|\langle X_T, G_T\rangle| > p\} \cap \{T < T_n\}$$

$$\subset \{|X|_T |G|_T > p\} \cap \{T < T_n\} \subset \{p < n|G_T|\} \cap \{T < T_n\} \subset \{p < nG_T\}$$

Since G is a uniformly integrable martingale, it is a martingale of class D; from $n|G_t| 1_{\{t<T_n\}} \leq n|G_t|$ we deduce that $n|G_t|1_{\{t<T_n\}}$ is a martingale of class (D):

$$\lim_{p\to\infty} \int_{\{n|G_t|1_{\{t<T_n\}}>p\}} n|G_t|1_{\{t<T_n\}} dP \leq \lim_{p\to\infty} \int_{\{n|G_t|>p\}} n|G_t| dP$$

$$= n \lim_{p\to\infty} \int_{\{|G_t|>\frac{p}{n}\}} n|G_t| dP = \lim_{\frac{p}{n}\to\infty} \int_{\{n|G_t|>p\}} n|G_t| dP = 0.$$

Hence there is a $p_{1\epsilon}$ such that for any $p \geq p_{1\epsilon}$ and any simple stopping time T we have

$$\int_{\{|(XG)_T^{T_n}|>p\}\cap\{T<T_n\}} n|(XG)_T^{T_n}| dP \leq \int_{\{n|G_t|>p\}} n|G_t| dP < \frac{\epsilon}{2} \tag{4}$$

We look now at the second term of the right hand side of the inequality (3).

$$\int_{\{|(XG)_T^{T_n}|>p\}\cap\{T\geq T_n\}} |(XG)_{T_n}| dP \leq \int_{\{|(XG)_{T_n}|>p\}} |(XG)_{T_n}| dP$$

Since $(XG)_{T_n} \in L^1$, for every $\epsilon > 0$ there is a $p_{2\epsilon} > 0$ such that for every $p \geq p_{2\epsilon}$ we have

$$\int_{\{|(XG)_T^{T_n}|>p\}} |(XG)_{T_n}| dP < \frac{\epsilon}{2} \tag{5}$$

If we put (4) and (5) together we deduce that for every $\epsilon > 0$ there is a $p_\epsilon = \max(p_{1\epsilon}, p_{2\epsilon})$ such that for any $p > p_\epsilon$ and any T simple stopping time we have

$$\int_{\{|(XG)_T^{T_n}|>p\}} |(XG)_T^{T_n}| dP < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the fact that $(XG)^{T_n}$ is a quasimartingale of class (D). From Theorem 14.2 of [Din00] we deduce that the Doléans measure $\mu_{(XG)^{T_n}}$ associated to the process $(XG)^{T_n}$ is $\sigma-$additive and has bounded variation on $\mathcal{R}$, hence it can be extended to a σ-additive measure with bounded variations on $\mathcal{P}$ (Theorem 7.4 b) of [Din00]).

Next we show that for any $B \in \mathcal{P}$ we have

$$\mu_{(XG)^{T_n}}(B) = \mu_{XG}(B \cap [0, T_n]).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$\mu_{(XG)^{T_n}}(\{0\} \times A) = \mu_{XG}((\{0\} \times A) \cap [0, T_n]).$$

and for $(s, t] \times A$ with $A \in \mathcal{F}_s$ we have

$$\mu_{(XG)^{T_n}}((s,t] \times A) = E(1_A((XG)_t^{T_n} - (XG)_s^{T_n})) = \mu_{XG}(((s,t] \times A) \cap [0, T_n]),$$

which proves our equality. Hence the measure μ_{XG} is σ-additive on the σ-ring $\mathcal{P} \cap [0, T_n]$ for each n, hence it is σ-additive on the ring

$$\mathcal{B} = \bigcup_{1 \leq n < \infty} \mathcal{P} \cap [0, T_n].$$

Next we observe that μ_{XG} is bounded on $\mathcal{R}$, therefore it has bounded variation on $\mathcal{R}$ which implies that the measure defined on $\mathcal{B} \cap \mathcal{R}$ is $\sigma-$additive and has bounded variation. Since $\mathcal{B} \cap \mathcal{R}$ generates $\mathcal{P}$, by Theorem 7.4 b) of [Din00], μ_{XG} can be extended to a σ-additive measure with bounded variation on $\mathcal{P}$.

Since $\langle I_X, g \rangle = \mu_{XG}$, it follows that $\langle I_X, g \rangle$ is bounded and σ-additive on $\mathcal{R}$, thus d) holds. The implication d)$\Longrightarrow$c) is evident.

a) $\Longrightarrow$ d) is evident since for each $g \in Z$, the measure $\langle I_X, g \rangle$ is σ-additive on $\mathcal{P}$ and since any $\sigma-$additive measure on a $\sigma-$algebra is bounded we conclude that for $g \in Z$, the measure $\langle I_X, g \rangle$ is bounded on $\mathcal{P}$ hence on $\mathcal{R}$.

Next we prove d)$\Longrightarrow$a). Assume d) is true. Then the real valued measure $\langle I_X, g\rangle$ is $\sigma-$additive and bounded on $\mathcal{R}$. Since we proved that b) $\Longleftrightarrow$ c) $\Longleftrightarrow$ d) we deduce from (1) that

$$|\langle I_X, g\rangle(A)| \leq M\|g\| \text{ for all } A \in \mathcal{R}$$

where $M = \sup\{|I_X(A)| : A \in \mathcal{R}\}$. By Proposition 2.16 of [Din00] it follows that the measure $\langle I_X(\cdot), g\rangle$ has bounded variation $|\langle I_X, g\rangle|(\cdot)$ satisfying

$$|\langle I_X, g\rangle|(A) \leq 2M\|g\|, \text{ for } A \in \mathcal{R}.$$

Applying Proposition 4.15 in [Din00] we deduce that $\tilde{I}_{X\mathbb{R},E}$ is bounded. By Theorem 3.7 b) of [BD01] we conclude that the measure $I_X : \mathcal{R} \to L^p_E$ has an additive extension $I_X : \mathcal{P} \to Z^{**}$ to $\mathcal{P}$ such that for each $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is a $\sigma-$additive on $\mathcal{P}$ which is Assertion a). $\square$

CHAPTER 5
EXAMPLES OF ADDITIVE SUMMABLE PROCESSES

In this chapter we present the definition and several properties of the processes with integrable variation and of the processes with integrable semivariation and we prove that they are additive summable. The fact that a process with integrable variation is summable is a simple, direct consequence of Theorem 19.13 in [Din00]. In Theorem 66 we show that the process X, with integrable semivariation is additive summable if the space G does not contain a copy of c_0. This proves that the class of additive summable processes is larger than the class of summable processes.

5.1 Examples of Additive Summable Processes

Definition 60. Let $X : \mathbb{R}_+ \times \Omega \to E$ be an E-valued process. We say that X has finite variation, if for each $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ has finite variation on each interval $[0, t]$. If $1 \leq p < \infty$, the process X has p-integrable variation if the total variation $|X|_\infty = \text{var}(X, \mathbb{R}_+)$ is p-integrable.

Definition 61. We define the variation process $|X|$ by

$$|X|_t(\omega) = var(X.(\omega), (-\infty, t]), \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$

where $X_t = 0$ for $t < 0$.

Noting that if $m : \mathcal{D} \to E \subset L(F, G)$ is a σ-additive measure then for each $z \in G^*$, the measure $m_z : \mathcal{D} \to F^*$ is $\sigma-$additive, we deduce that, if the process X is summable, then it is also additive summable. Hence the following theorem is a direct consequence of Theorem 19.13 in [Din00]

Theorem 62. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be a cadlag, adapted process with integrable variation $|X|$. Then X is 1–additive summable relative to any embedding $E \subset L(F, G)$.*

Proof. If $m : \mathcal{D} \to E \subset L(F,G)$ is a σ-additive measure then for each $z \in G^*$, the measure $m_z : \mathcal{D} \to F^*$ is $\sigma-$additive. We deduce that, if the process X is summable, then it is additive summable. Hence applying Theorem 19.13 b) in [Din00] we conclude our proof. $\square$

5.2 Processes with Integrable Semivariation

Definition 63. We define the semivariation process of X relative to (F,G) by

$$\tilde{X}_t(\omega) = svar_{F,G}(X_\cdot(\omega),\ (-\infty, t]),\ \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$

where $X_t = 0$ for $t < 0$.

Definition 64. The total semivariation of X is defined by

$$\tilde{X}_\infty(\omega) = \sup_{t \geq 0} \tilde{X}_t(\omega) = svar_{F,G}(X_\cdot(\omega), \mathbb{R}),\ \text{ for } \omega \in \Omega.$$

Definition 65. Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$. The process X is said to have finite semivariation relative to (F,G), if for every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ has finite semivariation relative to (F,G) on each interval $(-\infty, t]$. The process X is said to have p-integrable semivariation $\tilde{X}_{F,G}$ if the total semivariation $(\tilde{X}_{F,G})_\infty$ belongs to L^p.

Remark: If $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ is a process and $z \in G^*$ we define, the process $X_z : \mathbb{R}_+ \times \Omega \to F^*$ by

$$\langle x, (X_z)_t(\omega)\rangle = \langle X_t(\omega)x, z\rangle,\ \text{ for } x \in F, t \in \mathbb{R}_+ \text{ and } \omega \in \Omega.$$

For fixed $t \geq 0$, we consider the function $X_t : \omega \mapsto X_t(\omega)$ from Ω into $E \subset L(F,G)$ and for $z \in G^*$ we define $(X_t)_z : \Omega \to F^*$ by the equality

$$\langle x, (X_t)_z(\omega)\rangle = \langle X_t(\omega)x, z\rangle,\ \text{ for } \omega \in \Omega,\ \text{ and } x \in F.$$

It follows that

$$(X_t)_z(\omega) = (X_z)_t(\omega),\ \text{ for } t \in \mathbb{R}_+ \text{ and } \omega \in \Omega.$$

The semivariation $\tilde{X}$ can be computed in terms of the variation of the processes X_z:

$$\tilde{X}_t(\omega) = \sup_{z \in G_1^*} |X_z|_t(\omega).$$

If X has finite semivariation $\tilde{X}$, then each X_z has finite variation $|X_z|$.

The following theorem is an improvement over the Theorem 21.12 in [Din00], where it was supposed that $c_0 \notin E$ and $c_0 \notin G$.

Theorem 66. *Assume* $c_0 \not\subset G$. *Let* $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ *be a cadlag, adapted process with p-integrable semivariation relative to* $(\mathbb{R}, E)$ *and relative to* (F, G). *Then* X *is p-additive summable relative to* (F, G)

Proof. First we present the sketch of the proof, after which we prove all the details.

The prove goes as follows:

1) First we will show that

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{R} \text{ and } \omega \in \Omega, \tag{*}$$

where $A(\omega) = \{t; (t, \omega) \in A\}$ and $X(\omega)$ is $X_.(\omega)$. For the definition of the measure $m_{X(\omega)}$ see Section 2.2.

2) Then we will prove that the measure $m_{X(\omega)}$ has an additive extension to $\mathcal{B}(\mathbb{R}_+)$, with bounded semivariation relative to (F, G) and such that for every $g \in G^*$ the measure $(m_{X(\omega)})_g$ is $\sigma-$additive.

3) Next we prove that the function $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{P}$.

4) Then we show that the extension of the measure I_X to $\mathcal{P}$ has bounded semivariation relative to (F, L^p_G).

5) Finally we show that for each $z \in (L^p_G)^*$ the measure $(I_X)_z : \mathcal{P} \to F^*$ is $\sigma-$additive.

6) We conclude that the process X is $p-$additive summable.

Now we prove each step in detail.

1) First we prove (*) for predictable rectangles. Let $A = \{0\} \times B$ with $B \in \mathcal{F}_0$. Then we have

$$I_X(\{0\} \times B)(\omega) = 1_B(\omega)X_0(\omega) = \int 1_{\{0\}\times B}(s,\omega)dX_s(\omega) = m_{X(\omega)}(A(\omega)).$$

Now let $A = (s,t] \times B$ with $B \in \mathcal{F}_s$. In this case we also obtain

$$I_X((s,t] \times B)(\omega) = 1_B(\omega)(X_t(\omega) - X_s(\omega)) = \int 1_{(s,t]\times B}(p,\omega)dX_p(\omega) = m_{X(\omega)}(A(\omega)).$$

Since both $I_X(A)(\omega)$ and $m_{X(\omega)}(A(\omega))$ are additive we conclude that the equality (*) is true for $A \in \mathcal{R}$.

2) Since X has p-integrable semivariation relative to (F,G) we infer that $(\tilde{X}_{F,G})_\infty(\omega) < \infty$ a.s. If we redefine $X_t(\omega) = 0$ for those ω for which $(\tilde{X}_{F,G})_\infty(\omega) = \infty$ we obtain a process still denoted X with bounded semivariation. In this case for each $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is right continuous and with bounded semivariation. By Theorem 26 we deduce that the measure $m_{X(\omega)}$ can be extended to an additive measure $m_{X(\omega)} : \mathcal{B}(\mathbb{R}_+) \to E \subset L(F,G)$, with bounded semivariation relative to (F,G) and such that for every $g \in G^*$ the measure $(m_{X(\omega)})_g : \mathcal{B}(\mathbb{R}_+) \to F^*$ is $\sigma-$additive.

3) Since X has p-integrable semivariation relative to (F,G), for each $t \geq 0$ we have $X_t \in L^p_E$. Hence, by step 1, the function $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{R}$. To prove that $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{P}$ we will use the Monotone Class Theorem. We will prove that the set $\mathcal{P}_0$ of all sets $M \in \mathcal{P}$ for which the affirmation is true is a monotone class, containing $\mathcal{R}$, hence equal to $\mathcal{P}$. In fact, let M_n be a monotone sequence from $\mathcal{P}_0$ converging to M. By assumption, for each n the function $\omega \mapsto m_{X(\omega)}(M_n(\omega))$ belongs to L^p_E and for each ω the sequence $(M_n(\omega))$ is monotone in $\mathcal{B}(\mathbb{R}_+)$ and has limit $M(\omega)$. Moreover $|m_{X(\omega)}(M_n(\omega))| \leq \tilde{m}_{X(\omega)}(\mathbb{R}_+ \times \Omega) = \tilde{X}_\infty(\omega)$, which is p-integrable. By Lebesgue's

Theorem we deduce that the mapping $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L_E^p, hence $M \in \mathcal{P}_0$. Therefore $\mathcal{P}_0$ is a monotone class.

4) We use the equality (*) to extend I_X to the whole $\mathcal{P}$, by

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{P}.$$

Let $A \in \mathcal{P}$, $(A_i)_{i\in I}$ be a finite family of disjoint sets from $\mathcal{P}$ contained in A, and $(x_i)_{i\in I}$ a family of elements from F with $|x_i| \le 1$. Then we have

$$\begin{aligned} \|\sum I_X(A_i)x_i\|_p^p &= E(|\sum I_X(A_i)(\omega)x_i|^p) \\ &= E(|\sum m_{X(\omega)}(A_i(\omega))x_i|^p) \le E(|(\tilde{m}_{X(\omega)})_{F,G}(A(\omega))|^p) \\ &= \|(\tilde{m}_{X(\omega)})_{F,G}(A(\omega))\|_p^p = \|\tilde{X}_{F,G}(A(\omega))\|_p^p \le \|(\tilde{X}_{F,G})_\infty\|_p^p < \infty. \end{aligned}$$

Taking the supremum over all the families (A_i) and (x_i) as above, we deduce $(\tilde{I}_X)_{F,L_G^p} \le \|(\tilde{X}_{F,G})\|_p < \infty$.

5) Let $z \in (L_G^p)^*$ and $x \in F$. Then $z(\omega) \in G^*$ and for each set $M \in \mathcal{P}$ we have

$$\begin{aligned} \langle (I_X)_z(M), x\rangle &= \langle I_X(M)x, z\rangle = E(\langle I_X(M)(\omega)x, z(\omega)\rangle) \\ &= E(\langle m_{X(\omega)}(M(\omega))x, z(\omega)\rangle) = E(\langle (m_{X(\omega)})_{z(\omega)}(M(\omega)), x\rangle). \end{aligned} \tag{5.1}$$

By step we conclude that the measure $(I_X)_z$ is $\sigma-$additive for each $z \in (L_G^p)^*$.

6) By the definition in step 4,

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{P} \text{ and } \omega \in \Omega,$$

and by steps 2 and 3 we conclude that the measure I_X has an additive extension $I_X : \mathcal{P} \to L_E^p$. By step 5 the measure $(I_X)_z$ is $\sigma-$additive for each $z \in (L_G^p)^*$. By step 4 this extension has bounded semivariation. Therefore the process X is p-additive summable. □

The following theorem gives sufficient conditions for two processes to be indistinguishable. For the proof see [Din00], Corollary 21.10 b').

Theorem 67. *([Din00]21.10b')) Assume* $c_0 \not\subset E$ *and let* $A, B : \mathbb{R}_+ \times \Omega \to E$ *be two predictable processes with integrable semivariation relative to* $(\mathbb{R}, E)$*. If for every stopping time* T *we have* $E(A_\infty - A_T) = E(B_\infty - B_T)$*, then* A *and* B *are indistinguishable.*

The next theorem gives examples of processes with locally integrable variation or semivariation. For the proof see [Din00], Theorems 22.15 and 22.16.

Theorem 68. *([Din00]22.15,16) Assume* X *is right continuous and has finite variation* $|X|$ *(resp. finite semivariation* $\tilde{X}_{F,G}$*). If* X *is either predictable or a local martingale, then* X *has locally integrable variation* $|X|$ *(resp. locally integrable semivariation* $\tilde{X}_{F,G}$*).*

Proposition 69. *Let* $X : \mathbb{R}_+ \times \Omega \to E$ *be a process with finite variation. If* X *has locally integrable semivariation* $\tilde{X}_{\mathbb{R},E}$*, then* X *has locally integrable variation.*

Proof. Assume X has locally integrable semivariation $\tilde{X}$ relative to $(\mathbb{R}, E)$. Then there is an increasing sequence S_n of stopping times with $S_n \uparrow \infty$ such that $E(\tilde{X}_{S_n}) < \infty$ for each n. For each n define the stopping times T_n by $T_n = S_n \wedge \inf\{t \mid |X|_t \geq n\}$. It follows that $|X|_{T_n -} \leq n$. Since X has finite variation, by Proposition 6 we have $\Delta|X_{T_n}| = |\Delta X_{T_n}| \leq \tilde{X}_{T_n}$. From $\Delta|X|_{T_n} = |X|_{T_n} - |X|_{T_n -}$ we deduce that $|X|_{T_n} = |X|_{T_n -} + \Delta|X_{T_n}| \leq n + \tilde{X}_{T_n}$; Therefore $E(|X|_{T_n}) \leq n + E(\tilde{X}_{T_n}) < \infty$; hence X has locally integrable variation. □

5.3 Example of a Summable Process That is Not a Semimartingale

In this section we prove that in infinite dimensional Banach spaces, a locally summable process, is not necessarily a semimartingale, in the sense of definition given by Dellacherie and Meyer [DM78]. This follows from the Theorem 72, which states that under certain conditions on the Banach space E, a predictable semimartingale with finite semivariation has necessarily finite variation.

First we need to define the dual projection of a process.

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a right continuous, measurable process.

Definition 70. a) Assume that X has integrable variation $|X|$ (resp. $c_0 \not\subset E$ and X has integrable semivariation $\tilde{X}_{\mathbb{R},E}$). A right continuous, predictable process $Z : \mathbb{R}_+ \times \Omega \to E$ with integrable variation $|Z|$ (resp. with integrable semivariation $\tilde{Z}_{\mathbb{R},E}$) is called the predictable dual projection with integrable variation (resp. with integrable semivariation relative $(\mathbb{R}, E)$), if, for every real-valued, bounded, measurable process ϕ we have $E(\int \phi_s dZ_s) = E(\int {}^p\phi_s dX_s)$. where ${}^p\phi$ is the predictable projection of ϕ and the integrals are Lebesgue-Stieltjes integrals. We denote $Z = X^p$.

b) Assume X has locally integrable variation $|X|$ (resp. $c_0 \not\subset E$ and X has locally integrable semivariation $\tilde{X}_{\mathbb{R},E}$). A right continuous predictable process $Z : \mathbb{R}_+ \times \Omega \to E$ with locally integrable variation $|Z|$ (resp. with locally integrable semivariation $\tilde{Z}_{\mathbb{R},E}$) is called the predictable dual projection of X, if there is an increasing sequence (T_n) of stopping times with $T_n \uparrow \infty$, such that for each n, X^{T_n} and Z^{T_n} have integrable variation (resp. integrable semivariation relative $(\mathbb{R}, E)$) and Z^{T_n} is the predictable dual projection of X^{T_n}. We denote $Z = X^p$.

A similar definition is stated for the optional dual projection.

The existence of the dual projection is stated in the following theorem. For the proof see [Din00], Theorems 22.8 and 22.13.

Theorem 71. *Assume that E has the Radon-Nikodym Property and that X has integrable (resp. locally integrable) variation $|X|$. Then X has a predictable dual projection with integrable (resp. locally integrable) variation.*

Theorem 72. *Let E be a Banach space with the Radon-Nikodym property and with $c_0 \not\subset E$ (for example, E can be a reflexive space). Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a right continuous, predictable process with finite semivariation $\tilde{X}_{\mathbb{R},E}$.*

If X is a semimartingale, then X has locally integrable variation, outside an evanescent set.

Proof. We remark that since X is predictable and has finite semivariation $\tilde{X}_{\mathbb{R},E}$, by Theorem 68, X has locally integrable semivariation. $\tilde{X}_{\mathbb{R},E}$.

a) We assume first that X has integrable semivariation $\tilde{X}_{\mathbb{R},E}$ and that X is a semimartingale of the form $X = M + A$ where M is a right continuous, uniformly integrable martingale with integrable semivariation $\tilde{M}_{\mathbb{R},E}$ and A is a right continuous process with integrable variation $|A|$.

Let $z \in E^*$. Then $\langle X, z\rangle = \langle M, z\rangle + \langle A, z\rangle$. The processes $\langle X, z\rangle$, $\langle M, z\rangle$, $\langle A, z\rangle$ are adapted, right continuous and have integrable variation; $\langle X, z\rangle$ is predictable, and $\langle M, z\rangle$ is a uniformly integrable martingale. Consider the σ-additive stochastic measures with finite variation

$$\mu_{\langle X,z\rangle}, \mu_{\langle M,z\rangle}, \mu_{\langle A,z\rangle} : \mathcal{M} \to \mathbb{R},$$

defined for any real-valued, bounded, measurable process ϕ by

$$\int \phi d\mu_{\langle X,z\rangle} = E(\int \phi d\langle X, z\rangle),$$

$$\int \phi d\mu_{\langle M,z\rangle} = E(\int \phi d\langle M, z\rangle),$$

$$\int \phi d\mu_{\langle A,z\rangle} = E(\int \phi d\langle A, z\rangle).$$

(see [DM78], vi.64). Then $\mu_{\langle X,z\rangle} = \mu_{\langle M,z\rangle} + \mu_{\langle A,z\rangle}$. Since $\langle M, z\rangle$ is a uniformly integrable martingale with integrable variation, we have $\mu_{\langle M,z\rangle}(B) = 0$, for $B \in \mathcal{P}$. It follows that $\mu_{\langle X,z\rangle} = \mu_{\langle A,z\rangle}$ on $\mathcal{P}$. Let ϕ be a real-valued, bounded, measurable process and let ${}^p\phi$ be its predictable projection. Then

$$\int {}^p\phi d\mu_{\langle X,z\rangle} = E(\int {}^p\phi d\mu_{\langle A,z\rangle}), i.e.,$$

$$E(\int {}^p\phi d\langle X, z\rangle) = E(\int {}^p\phi d\langle A, z\rangle).$$

Since $\langle X, z\rangle$ is predictable, we have

$$E(\int {}^p\phi d\langle X, z\rangle) = E(\int \phi d\langle X, z\rangle).$$

It follows that

$$E(\int \phi d\langle X, z\rangle) = E(\int {}^p\phi d\langle A, z\rangle),$$

or

$$\langle E(\int \phi dX), z\rangle = \langle E(\int {}^p\phi dA), z\rangle,$$

where the integral are Stieltjes integral with respect to functions with finite semivariation.

Since $z \in E^*$ was arbitrary, we deduce that

$$E(\int \phi dX) = E(\int {}^p\phi dA).$$

It follows that X is the predictable dual projection with integrable semivariation $\tilde{X}_{\mathbb{R},E}$ of the process A with integrable semivariation $\tilde{A}_{\mathbb{R},E}$.

Since E has the Radon Nikodym Property and A has integrable variation, by Theorem 71, A has a predictable dual projection Y with integrable variation, satisfying,

$$E(\int \phi dY) = E(\int {}^p\phi dA),$$

where the integrals are Stieltjes integrals with respect to functions with finite variation, which are also Stieltjes integrals with respect to the same functions, considered with finite semivariation relative to $(\mathbb{R}, E)$. It follows that

$$E(\int \phi dX) = E(\int \phi dY).$$

If T is any stopping time, we take $\phi = 1_{(T,\infty)}$ and we obtain

$$E(X_\infty - X_T) = E(Y_\infty - Y_T).$$

Since both X and Y are predictable processes with integrable semivariation relative to $(\mathbb{R}, E)$, by Theorem 67, X and Y are indistinguishable. It follows that X has integrable variation except on an evanescent set.

b) Assume now that X is right continuous, predictable, with finite semivariation $\tilde{X}_{\mathbb{R},E}$ and that $X = M + A$ where M is a right continuous, local martingale and A is a right continuous process with finite variation.

From $M = X - A$ we deduce that M has finite semivariation $\tilde{M}_{\mathbb{R},E}$. Since $c_0 \not\subset E$, by Theorem 68, the martingale M has locally integrable semivariation $\tilde{M}_{\mathbb{R},E}$. Then, from $A = X - M$, it follows that A has locally integrable semivariation $\tilde{A}_{\mathbb{R},E}$; therefore by Proposition 69, A has locally integrable variation.

Let T_n be an increasing sequence of stopping times with $T_n \uparrow \infty$ such that for each n, X^{T_n} and M^{T_n} have integrable semivariation relative to $(\mathbb{R}, E), M^{T_n}$ is a uniformly integrable martingale and A^{T_n} has integrable variation. For each n we have $X^{T_n} = M^{T_n} + A^{T_n}$. By the first part of the proof, for each n, X^{T_n} has locally integrable variation, except on an evanescent set of the form $\mathbb{R}_+ \times N_n$ with $P(N_n) = 0$. The union $N = \cup_n N_n$ is P-negligible, and outside the evanescent set $\mathbb{R}_+ \times N$, each process X^{T_n} has integrable variation. It follows that X has locally integrable variation outside $\mathbb{R}_+ \times N$. □

We can deduce the following corollaries:

Corollary 73. *Let E be a Banach space with $c_0 \not\subset E$, having the RNP. Let $X : \mathbb{R}_+ \times \Omega \to E$ be a right continuous, predictable semimartingale. Assume that the set of points $\omega \in \Omega$ such that $X_.(\omega)$ has infinite variation is not negligible. Then the set of points $\omega \in \Omega$ such that $X_.(\omega)$ has infinite semivariation is not negligible.*

Corollary 74. *Let E be a Banach space with $c_0 \not\subset E$, having the RNP. Let $X : \mathbb{R}_+ \times \Omega \to E$ be a right continuous, predictable process with finite semivariation $\tilde{X}_{\mathbb{R},E}$. Assume that the set of points $\omega \in \Omega$ such that $X_.(\omega)$ has infinite variation is not negligible. Then X is not a semimartingale.*

Theorem 75. *Let E be an infinite dimensional Banach space with $c_0 \not\subset E$, having the RNP. There are E-valued summable processes which are not semimartingales.*

Proof. We shall construct a simple example of an $E-$valued, right continuous predictable, deterministic process with integrable semivariation relative to $(\mathbb{R}, E)$, having infinite variation. By Theorem 21.12 in [Din00] such a process is summable relative to $(\mathbb{R}, E)$. By Corollary 74, such a process cannot be a semimartingale. The construction of the example does not require the condition $c_0 \not\subset E$ or RNP.

Using the Dvoretzky-Rogers theorem, we can find a sequence $(x_n)_{1\leq n\leq\infty}$ of elements of E, such that the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent but not absolutely convergent.

Denote $s_0 = 0$, $s_1 = x_1, \cdots, s_n = \sum_{i=1}^{i=n} x_i, \cdots$.Then $x_n = s_n - s_{n-1}$, $\lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} x_i$ and $\sum_{n=1}^{\infty} |s_n - s_{n-1}| = \infty$.

Consider the function $g : \mathbb{R}_+ \to E$ defined by $g(0) = \sum_{n=1}^{\infty} x_n$ and $g(t) = \sum_{1\leq n<\infty} 1_{[\frac{1}{n+1},\frac{1}{n})}(t)s_n$, for $t > 0$. We define the deterministic process $X : \mathbb{R}_+ \times\Omega \to E$ by $X = g1_\Omega$, i.e., $X_t(\omega) = g(t)$ for $t \geq 0$ and $\omega \in \Omega$. We shall prove that X is a right continuous predictable process with finite, integrable semivariation $\tilde{X}_{\mathbb{R},E}$ and with infinite variation.

The proof is divided into several steps.

a) From the definition of g we deduce that g is right continuous; therefore X is right continuous.

b) g has infinite variation on $\mathbb{R}_+$; hence X has infinite variation on $\mathbb{R}_+$. In fact, we have $g(t) = s_n$ if $\frac{1}{n+1} \leq t < \frac{1}{n}$. We have then for each k,

$$\sum_{n=1}^{k} |g(\frac{1}{n}) - g(\frac{1}{n+1})| = \sum_{n=1}^{k} |s_{n-1} - s_n| = \sum_{n=1}^{k} |x_n|.$$

It follows that $var(g, \mathbb{R}_+) \geq \sup_k \sum_{n=1}^{k} |g(\frac{1}{n}) - g(\frac{1}{n+1})| = \sup_k \sum_{n=1}^{k} |x_n|$ $=\sum_{n=1}^{\infty} |x_n| = \infty$.

c) g has bounded semivariation $\tilde{g} = \tilde{g}_{\mathbb{R},E}$ on $\mathbb{R}_+$; In fact, let $x^* \in E^*$. The jumps of x^*g are at 0, equal to $x^*g(0)$ and at $\frac{1}{n}$, equal to $x^*s_{n-1} - x^*s_n$. Therefore $var(x^*g, \mathbb{R}_+) = |x^*g(0)| + \sum_{n=1}^{\infty} |x^*s_{n-1} - x^*s_n|$

$$\leq \sum_{n=1}^{\infty} |x^*(x_n)| + \sum_{n=1}^{\infty} |x^*(s_{n-1} - s_n)| = 2\sum_{n=1}^{\infty} |x^*(x_n)| < \infty.$$

Then, by Proposition 20.9 in [Din00] we have $svar_{\mathbb{R},E}(g, \mathbb{R}_+) < \infty$; hence the semivariation $\tilde{g}$ is bounded.

From $\tilde{X}_t(\omega) = \tilde{g}(t)$ we deduce that X has bounded semivariation $\tilde{X}$.

d) X has integrable semivariation $\tilde{X} = \tilde{X}_{\mathbb{R},E}$. In fact,

$$E(\tilde{X}_\infty) = \int \tilde{X}_\infty dP = \int svar_{\mathbb{R},E}(X, \mathbb{R}_+) dP = svar_{\mathbb{R},E}(g, \mathbb{R}_+) < \infty.$$

e) X is adapted since it is deterministic.

f) X is predictable since the sets $\{0\} \times \Omega$ and $[\frac{1}{n+1}, \frac{1}{n}) \times \Omega$ are predictable. □

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BIOGRAPHICAL SKETCH

I was born in Drobeta-Turnu-Severin, Romania, in 1974 and lived there until age 18. I received my secondary education in a School for the Arts (Şcoală de Arte), and in those years I took hundreds of hours of painting and art history. Also, by participating in the Mathematics Olimpiads every year I took the first steps in my career as a mathematician.

In 1992 I was admitted to "Facultatea de Matematică" at "Universitatea Bucureşti," which granted my "Licenţia" (Bachelor's degree) in 1997 with a specialization in Probability.

That same year I started graduate studies in Mathematics at the University of Florida under the guidance of Dr. N. Dinculeanu, and worked for the Department of Mathematics as a Teaching Assistant.

In the year 2000 I started working towards a Master's degree in Finance, in the Department of Finance and Real Estate of the University of Florida, under the supervision of Dr. M. Nimalendran.

Currently, I have accepted a post-doctoral position at Purdue University for the next three academic years.

ADDITIVE SUMMABLE PROCESSES AND THEIR STOCHASTIC INTEGRAL

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Degree: Doctor of Philosophy
Graduation Date: August 2002

In our work we define a class of processes, called additive summable, which is larger than the class of summable processes defined by Kussmaul, Brooks or Dinculeanu.

We also define the stochastic integral with respect to such a process and we prove that this integral verifies most of the important, measure theoretical, properties of the integral.

Eventually we give example of additive summable processes and prove that the stochastic integral we defined is more comprehensive than the classical stochastic integral defined with respect to a semimartingale.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

N. Dinculeanu

Nicolae Dinculeanu, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Murali Rao

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Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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